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Research article

Exploring the Foundational Significance of Gödel's Incompleteness Theorems

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Abstract:
Gödel's incompleteness theorems, published in 1931, are important and profound results in the foundations and philosophy of mathematics. On the basis of new advances in research on incompleteness in the literature, we discuss the correct interpretations of Gödel's incompleteness theorems, their influence on various fields, and the limit of their applicability. The motivation of this paper is threefold: to explore the foundational and philosophical significance of new advances in research on incompleteness since Gödel, to introduce new advances in research on incompleteness to the general philosophy community, and to commemorate the 90th anniversary of the publication of Gödel's incompleteness theorems.

Keywords:
The first incompleteness theorem, The second incompleteness theorem, The limit of incompleteness, The intensional problem, Interpretation

1. Introduction

Gödel's incompleteness theorems published in 1931 [Gödel (1931)] are important and profound results in the foundations and philosophy of mathematics and have had a wide influence on the development of logic, philosophy, mathematics, computer science, and other fields.

Good textbooks and survey papers have been written about Gödel's incompleteness theorems. For textbooks, refer to Boolos (1993), Enderton (2001), Lindström (1997), Murawski

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(1999), Smith (2007), and Smullyan (1992). Since Gödel, there have been many advances in the study of Gödel's incompleteness theorems. For survey papers, refer to Beklemishev (2010), Cheng (2021), Kotlarski (2004), Smoryński (1977), and Visser (2016).

The motivation of this paper is threefold: to explore the foundational significance of the research on incompleteness since Gödel; to introduce new advances in research on incompleteness to the general philosophy community; and to commemorate the 90th anniversary of the publication of Gödel's incompleteness theorems. On the basis of new advances in research on incompleteness in the literature, we discuss correct interpretations of Gödel's incompleteness theorems, their influence on various fields, and the limit of their applicability. In this paper, we focus on the foundational and philosophical significance of these advances in research on incompleteness in the last 30 years.

The structure of this paper is as follows. In Section 1, we introduce the motivation of this paper and its structure. In Section 2, we present Gödel's incompleteness theorems and their modern versions, discuss the main ideas of Gödel's proof, give some explanations of the incompleteness theorems, and give a summary of different proofs of Gödel's theorems. In Section 3, we discuss the influence of the incompleteness theorems on the foundations of mathematics, ordinary mathematics, theoretic computer science, and philosophy. In Section 4, we discuss the limit of the applicability of Gödel's first and second incompleteness theorems. In Section 5, we conclude the paper.

2. Interpretation of Gödel's Incompleteness Theorems

The structure of this section is as follows. In Section 2.1, we list some basic definitions and facts used in this paper. In Section 2.2, we present Gödel's original incompleteness theorems and their modern variants. In Section 2.3, we discuss the main ideas of Gödel's proof of his incompleteness theorems. In Section 2.4, we give some explanations of the incompleteness theorems. In Section 2.5, we give a summary of different proofs of Gödel's theorems.

2.1 Preliminaries

We first review some basic definitions used in this paper, which are standard in the literature. A theory consists of three basic components: a formal language, axioms, and deduction rules. A formal language includes a symbol list and the formation rules of formulas. The symbol list includes logical and non-logical symbols. The formulas are recursively defined according to the formation rules of formulas. The axioms are a special kind of formulas (including logical axioms and non-logical axioms) that are the starting point of a deduction. A rule of deduction tells us how to derive a formula from some given formulas.

For a given theory T, we use $L(T)$ to denote the language of T and often equate $L(T)$ with the list of non-logical symbols of the language. For a formula ϕ in $L(T)$, $T \vdash \phi$ denotes that ϕ is provable in T, i.e., there is a finite sequence of formulas $\langle \phi_0, \ldots, \phi_n \rangle$ such that $\phi_n = \phi$, and for any $0 \leq i \leq n$, either ϕ_i is an axiom of T or ϕ_i follows from some $\phi_j (j < i)$ by using one inference rule; otherwise, we say ϕ is not provable in T. We say a theory T is consistent if no contradiction is provable in T. We say a sentence ϕ is independent of T if both ϕ and $\neg\phi$ are not provable in T. We say a theory T is complete if for any sentence ϕ in $L(T)$, either $T \vdash \phi$ or $T \vdash \neg\phi$; otherwise, T is incomplete (i.e., there is a sentence ϕ in $L(T)$ that is independent of T). Intuitively, we say a theory T is recursively axiomatized if there is an effective algorithm such that for any formula A in $L(T)$, this algorithm can decide whether A is an axiom of theory T. We say a theory T is essentially incomplete if any recursively axiomatized consistent extension in $L(T)$ of T is incomplete.

Robinson Arithmetic **Q** was introduced by Tarski, Mostowski, and Robinson (1953) as a base axiomatic theory for investigating incompleteness and undecidability. Robinson Arithmetic **Q** is defined in the language $\{\mathbf{0}, \mathbf{S}, +, \times\}$ with the following axioms:

$\mathbf{Q}_1$ $\forall x \forall y(\mathbf{S}x = \mathbf{S}y \rightarrow x = y)$;

$\mathbf{Q}_2$ $\forall x(\mathbf{S}x \neq \mathbf{0})$;

$\mathbf{Q}_3$ $\forall x(x \neq \mathbf{0} \rightarrow \exists y(x = \mathbf{S}y))$;

$\mathbf{Q}_4$ $\forall x(x + \mathbf{0} = x)$;

$\mathbf{Q}_5$ $\forall x \forall y(x + \mathbf{S}y = \mathbf{S}(x + y))$;

$\mathbf{Q}_6$ $\forall x(x \times \mathbf{0} = \mathbf{0})$;

$\mathbf{Q}_7$ $\forall x \forall y(x \times \mathbf{S}y = x \times y + x)$.

Peano Arithmetic **PA** has the same language as **Q** and consists of axioms $\mathbf{Q}_1$−$\mathbf{Q}_2$, $\mathbf{Q}_4$−$\mathbf{Q}_7$ in the above definition of **Q** and the following axiom scheme of induction: $(\phi(0) \wedge \forall \boldsymbol{x}(\phi(\boldsymbol{x}) \rightarrow \phi(\mathbf{S}\boldsymbol{x}))) \rightarrow \forall \boldsymbol{x}\phi(\boldsymbol{x})$, where ϕ is a formula with at least one free variable $\boldsymbol{x}$.

The language $L(\mathbf{PA})$ is called arithmetic language. In the standard model of arithmetic (denoted by $\mathfrak{N}$), the domain of $\mathfrak{N}$ is the set $\mathbb{N}$ of natural numbers, the symbol $\mathbf{0}$ is interpreted as the least number 0, the function symbol $\mathbf{S}$ is interpreted as the successor function on $\mathbb{N}$: $S(n) = n + 1$, and the function symbols $+$ and $\times$ are respectively interpreted as the addition and multiplication operations on $\mathbb{N}$. Any natural number n is represented by a term $\overline{n}$ in $L(\mathbf{PA})$.[1]

Arithmetization is an important method in mathematical logic. The main idea of arithmetization is to code each symbol in the language of a theory by a natural number. Since we can code a finite sequence of natural numbers by a single natural number, relations and operations on strings can be transformed into relations and operations on natural numbers.[2] Under the arithmetization, any formula or sequence of formulas can be coded by a natural number (called a Gödel number). Given a formula in the arithmetic language, we use $\ulcorner\phi\urcorner$ to denote the term representing the Gödel number of ϕ. In this paper, we equate a set of formulas with the set of Gödel numbers of these formulas; whenever we say a set of formula is recursive, it means the corresponding set of Gödel numbers is recursive.

We introduce a hierarchy of $L(\mathbf{PA})$ formulas called the arithmetical hierarchy [see Hájek & Pudlák (1993)]. Bounded formulas (Σ_0^0, Π_0^0, or Δ_0^0 formulas) are built from atomic formulas using only propositional connectives and bounded quantifiers (in the form $\forall \boldsymbol{x} \leq \boldsymbol{y}$ or $\exists \boldsymbol{x} \leq \boldsymbol{y}$). We say a formula is Σ_{n+1}^0 if it is equivalent to a formula in the form $\exists \boldsymbol{x}\phi$, where ϕ is Π_n^0; a formula is Π_{n+1}^0 if it is equivalent to a formula in the form $\forall \boldsymbol{x}\phi$, where ϕ is Σ_n^0.[3] A formula is Δ_n^0 if it is equivalent to both a Σ_n^0 formula and a Π_n^0 formula in **PA**. We say a theory T is $\boldsymbol{\omega}$-consistent if there is no formula $\varphi(x)$ such that $T \vdash \exists x\varphi(x)$ and for any $n \in \mathbb{N}$, $T \vdash \neg\varphi(\overline{n})$; T is 1-consistent if there is no such Δ_1^0 formula $\varphi(x)$.

The fixed-point lemma says that if T is a recursively axiomatized consistent extension of **Q**, then for any formula $\phi(x)$ with only one free variable, there is a sentence θ such that $T \vdash \theta \leftrightarrow \phi(\ulcorner\theta\urcorner)$.

2.2 Formulation of Gödel's incompleteness theorems

In 1931, Gödel proved his first incompleteness theorem for a certain formal system **P** related to Whitehead and Russell's *Principia Mathematica* based on the simple theory of types over the natural number series and the Dedekind–Peano axioms [see Beklemishev (2010, p. 3)]. Gödel's original first incompleteness theorem [Theorem XI in Gödel (1931)] says that for any recursively axiomatized extension T of **P** in the same language as **P**, if T is

1 For any $n \in \mathbb{N}$, we define the term $\overline{n}$ representing n as $\overline{0} = \mathbf{0}$ and $\overline{n+1} = \mathbf{S}\overline{n}$.

2 For technical details of arithmetization, refer to Murawski (1999).

3 Thus, a Σ_n^0-formula has a block of n alternating quantifiers, the first one being existential, and this block is followed by a bounded formula. The definition for Π_n^0-formulas is similar.

ω-consistent, then *T* is incomplete. Theorem VI of Gödel (1931) contains his second incompleteness theorem: if *T* is a recursively axiomatized consistent extension of **P**, then the consistency of *T* is not provable in *T*, where the consistency of *T* is formulated as an arithmetic formula saying that there exists an unprovable sentence in *T*. In a footnote of Gödel (1931), he comments that the second incompleteness theorem is a corollary of the first incompleteness theorem (and, in fact, a formalized version of the first incompleteness theorem). In Gödel (1931), he sketches a proof of his second incompleteness theorem and promises to provide full details in a subsequent publication. However, the follow-up paper was never published, and a detailed proof of the second incompleteness theorem for first-order arithmetic first appeared in a monograph by Hilbert and Bernays (1939).

In Gödel (1932), Gödel formulates his incompleteness theorems for extensions of a variant of **PA**. In Gödel (1934), he gives another treatment of his results in Gödel (1931). In 1963, in a footnote of the English translation of his 1931 paper, Gödel writes [Friedman (2009)]: "In consequence of later advances, in particular of the fact that due to A.M. Turing's work, a precise and unquestionably adequate definition of the general notion of a formal system can now be given, a completely general version of Theorems VI and XI is now possible. That is, it can be proved rigorously that in every consistent formal system that contains a certain amount of finitary number theory, there exist undecidable arithmetic propositions and that, moreover, the consistency of any such system cannot be proved in the system."

Let *T* be a recursively axiomatized extension of **Q**. The following is a modern reformulation of Gödel's incompleteness theorems.

> [Gödel's first incompleteness theorem]: If *T* is ω-consistent, then *T* is incomplete.
>
> [Gödel's second incompleteness theorem]: If *T* is consistent, then the consistency of *T* is not provable in *T*.

Rosser (1936) improves Gödel's first incompleteness theorem as follows, only assuming the consistency of *T*:

> [Gödel–Rosser first incompleteness theorem]: If *T* is a recursively axiomatized consistent extension of **Q**, then *T* is incomplete.

In this paper, we freely use G1 to refer to the Gödel–Rosser first incompleteness theorem and its variants, and use G2 to refer to Gödel's second incompleteness theorem and its variants. The meanings of G1 and G2 in this paper will be clear from the context.

2.3 Main ideas of Gödel's proof

In this section, we outline the main ideas of Gödel's proof of his incompleteness theorems and sketch a standard proof of Gödel's theorems. We first explain some key notions in Gödel's theorems: recursively axiomatized, complete, and ω-consistent. If otherwise specified, we assume that T is a recursively axiomatized consistent extension of **Q**.

Firstly, the condition that "the theory T is recursively axiomatized" is necessary for G1. A non-recursively axiomatized extension of **Q** may be complete. In the philosophy literature, a popular interpretation of Gödel's incompleteness theorems is that any consistent extension of **Q** is incomplete. In fact, this interpretation is incorrect: not all consistent extensions of **Q** are incomplete. For example, considering the set of sentences in $L(\mathbf{PA})$ that are true in the standard model of arithmetic, this theory is a consistent extension of **Q** but it is complete. The point is that this theory is not recursively axiomatized, and hence G1 does not apply to this theory.

Secondly, we should distinguish the notions of semantic completeness and syntactic completeness. The notion of "complete" in G1 refers to syntactic completeness. We say a theory T is semantically complete if for any sentence A in $L(T)$, $T \vDash A$ if and only if $T \vdash A$. From Gödel's completeness theorem of first-order logic, any first-order theory is semantically complete. However, it is not true that any first-order theory is syntactically complete.

Thirdly, Gödel's first incompleteness theorem assumes that theory T is ω-consistent, which is stronger than "T is consistent". All ω-consistent theories are consistent, but the converse does not hold: there are consistent theories that are not ω-consistent.

The three main ideas in Gödel's proof of his incompleteness theorems are arithmetization, representability, and self-reference. Firstly, via arithmetization, we can translate metamathematical statements of a formal theory T into statements about natural numbers. Furthermore, fundamental metamathematical relations can be translated in this way into certain recursive relations, and hence into relations representable in T since Gödel proves that every recursive relation is representable in **PA**.[4] Consequently, one can speak about a formal theory of arithmetic and its properties in the system itself! This is the essence of Gödel's idea of arithmetization.

Secondly, we can define certain relations on natural numbers that express crucial metamathematical concepts related to a theory T, such as "proof" and "consistency". For example, we can define a binary relation on $\mathbb{N}^2$ as follows: $Proof_T(m, n)$ if and only if n is the

4 Take a 1-ary relation $R(x)$ on $\mathbb{N}$ for example. We say $R(x)$ is representable in T if there exists a formula $\phi(x)$ with only one free variable x such that if $R(m)$ holds, then $T \vdash \phi(\overline{m})$; if $R(m)$ does not hold, then $T \vdash \neg\phi(\overline{m})$. We say $\phi(x)$ is the formula representing $R(x)$.

Gödel number of a proof in T of the formula with Gödel number m. We can show that the relation $Proof_T(m, n)$ is recursive. Next, let $\mathbf{Prf}_T(x, y)$ be the formula representing $Proof_T(m, n)$ in **PA**. Via arithmetization and representability, one can speak about the property of T in **PA** itself! From the formula $\mathbf{Prf}_T(x, y)$, we can define the provability predicate $\mathbf{Pr}_T(x)$ as $\exists y \mathbf{Prf}_T(x, y)$.

Then, Gödel defined the Gödel sentence **G**, which asserts its own unprovability in T via a self-reference construction. One way of obtaining the Gödel sentence is to use the fixed-point lemma, which implies that the predicate $\neg\mathbf{Pr}_T(x)$ has a fixed point, i.e., there is a sentence **G** such that $T \vdash \mathbf{G} \leftrightarrow \neg\mathbf{Pr}_T(\ulcorner \mathbf{G} \urcorner)$. Gödel showed that if T is consistent, then $T \nvdash \mathbf{G}$, and if T is ω-consistent, then $T \nvdash \neg\mathbf{G}$. Thus, if T is ω-consistent, then **G** is independent of T, and hence T is incomplete.

We say a formula $\mathbf{Pr}_T(x)$ is a standard provability predicate if it satisfies the following conditions:

D1: If $T \vdash \phi$, then $T \vdash \mathbf{Pr}_T(\ulcorner \phi \urcorner)$;

D2: $T \vdash \mathbf{Pr}_T(\ulcorner \phi \rightarrow \varphi \urcorner) \rightarrow (\mathbf{Pr}_T(\ulcorner \phi \urcorner) \rightarrow \mathbf{Pr}_T(\ulcorner \varphi \urcorner))$;

D3: $T \vdash \mathbf{Pr}_T(\ulcorner \phi \urcorner) \rightarrow \mathbf{Pr}_T(\ulcorner \mathbf{Pr}_T(\ulcorner \phi \urcorner) \urcorner)$.

The provability predicate $\mathbf{Pr}_T(x)$ that Gödel constructs satisfies conditions D1–D3, and hence is a standard provability predicate.[5]

From provability predicate $\mathbf{Pr}_T(x)$, we can define the consistency statement $\mathbf{Con}(T) \triangleq \neg\mathbf{Pr}_T(\ulcorner \mathbf{0} \neq \mathbf{0} \urcorner)$, an arithmetic sentence expressing the consistency of T that says that a contradiction such as $\mathbf{0} \neq \mathbf{0}$ is not provable in T. We say $\mathbf{Con}(T)$ is the canonical consistency statement if it is defined via a standard provability predicate $\mathbf{Pr}_T(x)$. In this paper, unless otherwise stated, we assume that $\mathbf{Pr}_T(x)$ is a standard provability predicate, and we use the canonical consistency statement to express the consistency of theory T.

From properties D1–D3 of the standard provability predicate $\mathbf{Pr}_T(x)$, we can prove that $T \vdash \mathbf{G} \leftrightarrow \mathbf{Con}(T)$. Thus, G2 holds: if T is consistent, then $T \nvdash \mathbf{Con}(T)$.

5 In Rosser's proof of G1, he uses the Rosser provability predicate instead, which is defined as follows: $\mathbf{Pr}_T^R(x) \triangleq \exists y(\mathbf{Prf}_T(x, y) \wedge \forall z \leq y \neg\mathbf{Prf}_T(\dot{\neg} x, z))$, where $\mathbf{Prf}_T(x, y)$ is a proof predicate. The Rosser provability predicate is not standard.

2.4 Some explanations of Gödel's incompleteness theorems

We first give some explanations of Gödel's first incompleteness theorem. Firstly, G1 does not tell us that any consistent first-order theory is incomplete. In fact, there are many consistent complete first-order theories. For example, the theory of dense linear orderings without endpoints (**DLO**) is complete. Even for arithmetic theories, not all of them are incomplete.[6]

Secondly, Gödel avoids the use of semantic notions such as model, truth, and definability in his original statement of the incompleteness theorems. On the one hand, before Gödel's paper in 1931, there was considered to be no clear difference between the notion of truth and the notion of provability. On the other hand, owing to the existence of semantic paradoxes in logic, semantic notions were under suspicion in Gödel's time [see Beklemishev (2010)]. Tarski's formal definition of truth in a formal language was not published until 1933, which explains why Gödel uses syntactic notions such as ω-consistency instead of semantic notions in his incompleteness theorems. In 1931, there was still no precise notion of the interpretation of one theory in another theory. Thus, the original statement of Gödel's incompleteness theorems was confined to the complex theory **P** and its language.

Thirdly, in Gödel's proof of his first incompleteness theorem, we can show that Gödel sentence **G** is not provable in T only by assuming that T is consistent. However, only assuming that T is consistent is insufficient to show that ¬**G** is also not provable in T. To show that **G** is independent of T, we need to assume a stronger condition than the consistency of T. Gödel proves that **G** is independent of T if T is ω-consistent.[7]

Fourthly, a complete theory may be incomplete after deleting some axioms, and an incomplete theory may be complete after adding some axioms. However, what G1 expresses is the essential incompleteness of **Q**: not only is **Q** incomplete, but also any recursively axiomatized consistent extension of **Q** is incomplete (i.e., regardless of how we effectively add new axioms to **Q**, the new consistent extension of **Q** always has independent statements).

Fifthly, the notion of provability used in Gödel's incompleteness theorems is relative provability (provability relative to a theory), not absolute provability. Gödel's incompleteness theorems do not tell us that some truth is absolutely unprovable. From G1, any recursively axiomatized consistent extension T_0 of **Q** has some independent statement A_0. However, A_0 is provable in some recursively axiomatized consistent extension T_1 of T_0. From G1, T_1 also has

6 For example, arithmetic theories (theories in the arithmetic language) such as the theory of algebraically closed fields of a given characteristic ($\mathbf{ACF_p}$) and the theory of real closed fields (**RCF**) are all complete.

7 In fact, the optimal condition to show that the Gödel sentence is independent of T is that $T + \mathbf{Con}(T)$ is consistent [see Theorems 35 and 36 in Isaacson (2011)].

some independent statement A_1. Similarly, A_1 is provable in some recursively axiomatized consistent extension T_2 of T_1, but T_2 also has some independent statement A_2. This process can continue forever.

Let T be a recursively axiomatized consistent extension of **Q**. Sixthly, even if T is incomplete, all Σ^0_1 sentences true in the standard model of arithmetic are provable in T. However, this does not hold for Π^0_1 sentences: there always exists a Π^0_1 true sentence unprovable in T. For example, Gödel's sentence is a Π^0_1 true sentence in the form $\forall x\phi(x)$ that is not provable in T, but for any natural number n, $T \vdash \phi(\overline{n})$. Turing (1939) showed that any Π^0_1 true sentence is provable in some transfinite iterated extension of **PA**. Feferman (1962) extended Turing's work and proved that any true arithmetic sentence is provable in some transfinite iterated extension of **PA**.

Now we give some explanations of G2. Firstly, even if **Con**(**PA**) is not provable in **PA** by G2, for any finite sub-theory S of **PA**, **Con**(S) is provable in **PA**.

Secondly, a common misinterpretation of G2 is that the consistency of **PA** can only be proved in some extension of **PA**. Gentzen (1936) proposed a theory (called Gentzen's theory) that is primitive recursive arithmetic with the additional principle of quantifier-free transfinite induction up to the ordinal ϵ_0.[8] Gentzen proved the consistency of **PA** in Gentzen's theory. Gentzen's theory does not contain **PA** and is not contained in **PA**, but it is stronger than **PA** w.r.t. interpretation.

Thirdly, G2 only tells us that if T is consistent, then **Con**(T) is not provable in T.[9] However, G2 does not tell us that if T is consistent, then **Con**(T) is independent of T. Only assuming that T is consistent is insufficient to show that $T \nvdash \neg\mathbf{Con}(T)$. To show that **Con**($T$) is independent of T, we need to assume stronger conditions. In fact, assuming that T is 1-consistent, we can show that **Con**(T) is independent of T. Note that "T is 1-consistent" is stronger than "T is consistent".

Fourthly, the differences between Gödel sentence and Rosser sentence and between **Con**(T) and $1-\mathbf{Con}(T)$ are very important. However, these differences are often overlooked in informal philosophical discussions of Gödel's incompleteness theorems. Let $1-\mathbf{Con}(T)$ be the sentence in L(**PA**) expressing that T is 1-consistent and T be a recursively axiomatized consistent extension of **PA**. We have the following facts:

(1) $T \vdash \mathbf{Con}(T) \rightarrow \mathbf{Con}(T + \neg\mathbf{Con}(T))$;

8 ϵ_0 is the least ordinal α such that $\alpha = \omega^{\alpha}$.

9 Recall that $\mathbf{Con}(T) = \neg\mathbf{Pr}_T(\ulcorner \mathbf{0} \neq \mathbf{0} \urcorner)$, where $\mathbf{Pr}_T(x)$ is a standard provability predicate.

(2) $T \nvdash \mathbf{Con}(T) \to \mathbf{Con}(T + \mathbf{Con}(T))$;

(3) $T \vdash \mathbf{Con}(T) \to \mathbf{Con}(T + \theta)$ where θ is the Rosser sentence;[10]

(4) $T \vdash 1 - \mathbf{Con}(T) \to \mathbf{Con}(T + \mathbf{Con}(T))$.

A misinterpretation of the application of G2 is that we can iteratively add the consistency statement: from $\mathbf{Con}(T)$, we have $\mathbf{Con}(T + \mathbf{Con}(T))$ and then $\mathbf{Con}(T + \mathbf{Con}(T + \mathbf{Con}(T)))$, and we can continue forever. However, from the facts above, this does not hold. To iterate the consistency statement, we need to assume a condition stronger than "T is consistent": T is 1-consistent. Assuming that $1 - \mathbf{Con}(T)$ holds, then we have $\mathbf{Con}(T)$, $\mathbf{Con}(T + \mathbf{Con}(T))$, and then $\mathbf{Con}(T + \mathbf{Con}(T + \mathbf{Con}(T)))$, and we can continue forever (in fact, $1 - \mathbf{Con}(T)$ is stronger than all these sentences).[11]

Fifthly, both mathematically and philosophically, G1 and G2 are of a rather different nature and scope. The consistency statement is obviously of a logical nature rather than of a mathematical nature. The meaning of G1 does not depend on arithmetization and the use of a provability predicate. In this sense, we can say that G1 is extensional. However, the meaning of G2 depends on how to express the consistency statement. In this sense, G2 is intensional and whether G2 holds depends on many factors. Refer to Section 4 for more discussion about the intensionality problem of G2.

2.5 Different proofs of Gödel's incompleteness theorems

Since Gödel, many different proofs of Gödel's incompleteness theorems have been found. For the proof of G1, it is sufficient to show that there exists an independent sentence, regardless of whether it is constructed via methods from ordinary mathematics or from pure logic. From the various proofs of Gödel's incompleteness theorems in the literature, we can classify these different proofs on the basis of the following nine properties: (1) proof-theoretic proof, (2) recursion-theoretic proof, (3) model-theoretic proof, (4) proof via arithmetization, (5) proof via the fixed-point lemma, (6) proof based on "logical paradox", (7) constructive

10 Let $\mathbf{Pr}_T^R(x)$ be a Rosser predicate. The Rosser sentence of T is the fixed point of $\neg\mathbf{Pr}_T^R(x)$ when we apply the fixed-point lemma to the predicate $\neg\mathbf{Pr}_T^R(x)$, i.e., $T \vdash \theta \leftrightarrow \neg\mathbf{Pr}_T^R(\ulcorner\theta\urcorner)$. We can show that if T is consistent, then the Rosser sentence θ is independent of T.

11 See Pudlák (1999).

proof,[12] (8) proof having the Rosser property,[13] and (9) independent sentence having real mathematical content.[14] Gödel's proof of his first incompleteness theorem has the following properties: it uses a proof-theoretic method with arithmetization; it does not directly use the fixed-point lemma; the proof is constructive; the proof does not have the Rosser property; Gödel's sentence has no real mathematical content.[15] These different proofs of Gödel's incompleteness theorems reveal the relationship among distinct fields, including proof theory, computability theory, model theory, logical paradox, and ordinary mathematics. For a detailed discussion of different proofs of Gödel's theorems, refer to Cheng (2021).

Now, we give some comments on these different proofs. Firstly, the above properties (1)–(9) only emphasize one aspect of proofs of Gödel's incompleteness theorems and they are not exclusive: a proof of G1 or G2 may satisfy several of the above properties.

Secondly, none of the above properties is a necessary condition for proving the incompleteness theorems. For any of the above properties, we can find examples of proofs with this property and also examples of proofs without this property in the literature. For example, G1 also has non-constructive proofs, proofs without the use of logical paradox, proofs with the Rosser property, proofs without the use of arithmetization, and so forth.

Thirdly, most proofs of G1 and G2 use arithmetization. However, Grzegorczyk (2005) proposed **TC** as an alternative theory for studying incompleteness and undecidability, and showed that **TC** is essentially incomplete and mutually interpretable with **Q** without the use of arithmetization.

Fourthly, Gödel's incompleteness theorems are closely related to logical paradox. In the literature, there are many proofs of Gödel's incompleteness theorems based on logical paradox. Gödel comments that "any epistemological antinomy could be used for a similar proof of the existence of undecidable propositions". We can view Gödel's sentence as some variant of the liar sentence in the liar paradox. Gödel's sentence concerns the notion of provability but the liar sentence concerns the notion of truth. There is a major difference

12 We say a proof of G1 is constructive if it explicitly constructs the independent sentence of the base theory by algorithmic means.

13 We say a proof has the Rosser property if it only assumes that the base theory is consistent instead of assuming that the base theory is ω-consistent.

14 i.e., the asserted independent sentence arises from ordinary mathematics and is not a purely logical construction.

15 i.e., Gödel's sentence is a purely logical construction (via the arithmetization of syntax and the provability predicate) and has no relevance to ordinary mathematics (without any combinatorial or number-theoretic content). On the contrary, the Paris–Harrington principle is an independent arithmetic sentence from ordinary mathematics with combinatorial content.

between Gödel's sentence and the liar sentence. Gödel's sentence does not lead to a contradiction as the liar sentence does. In addition to the liar paradox, many other paradoxes have been used to give new proofs of incompleteness theorems: for example, Berry's paradox, the Grelling–Nelson paradox, the unexpected examination paradox, and Yablo's paradox [see Cheng (2021)]. The key point of proofs of Gödel's incompleteness theorems based on logical paradox is to properly formalize the logical paradox used. These distinct proofs of Gödel's theorems based on logical paradox confirm Gödel's view on the connection between incompleteness and logical paradox.

Fifthly, most proofs of Gödel's incompleteness theorems use purely logical methods, and the independent sentence constructed has a clear logical flavor without real mathematical content. Since Gödel, many independent sentences from ordinary mathematics with real mathematical content have been found. These sentences have a clear mathematical flavor and do not refer to the arithmetization of syntax and provability. Paris and Harrington (1977) proposed the Paris–Harrington principle from ordinary mathematics with combinatorial content and proved that it is independent of **PA**. For more examples of independent sentences of **PA** from ordinary mathematics, refer to Bovykin (2006), Cheng (2019, 2021), and Friedman (2022).

3. Impact of Gödel's Incompleteness Theorems

Gödel's incompleteness theorems have had wide, deep, and lasting influence on logic, philosophy, ordinary mathematics, and theoretic computer science. In this section, on the basis of the research on incompleteness theorems in the literature, we analyze the influence of Gödel's incompleteness theorems on logic (especially the foundations of mathematics), ordinary mathematics, theoretic computer science, and philosophy. In fact, the influence of Gödel's theorems is not confined to these fields. Because of the space limit, we only discuss some of the main influences of Gödel's incompleteness theorems on these four fields.

3.1 Impact of Gödel's theorems on the foundations of mathematics

The influence of Gödel's incompleteness theorems on the foundations of mathematics is mainly reflected in the following aspects. Firstly, in some sense, Gödel's incompleteness theorems reveal the essential limitation of the formal method in logic. Gödel's incompleteness theorems show the limit of provability of any recursively axiomatized consistent theory extending or interpreting **Q**: regardless of the strength of theory T, if T is a recursively axiomatized consistent theory containing enough information about arithmetic such as **Q**, there always exists a statement independent of *T*. Robinson arithmetic **Q** is a very weak arithmetic theory. Thus, we can say that the incompleteness theorems reveal the limit of provability of most formal theories.

Secondly, Gödel's incompleteness theorems reveal the essential difference between the notion of truth and the notion of provability. Before Gödel's incompleteness theorems, it was thought that "a proposition is true" equals "a proposition is provable". Gödel's incompleteness theorems show the difference between "provable in **PA**" and "true in the standard model of arithmetic": all arithmetic sentences provable in **PA** are true in the standard model of arithmetic, but there are true arithmetic sentences independent of **PA**. As an application of the fixed-point lemma, Tarski's theorem on the undefinability of truth tells us that the set of sentences true in the standard model $\mathfrak{N}$ of arithmetic is not definable in $\mathfrak{N}$. However, the set of sentences provable in **PA** is definable in $\mathfrak{N}$ even if it is not recursive.

Thirdly, Gödel's incompleteness theorems refute a strong version of Whitehead and Russell's program of logicism, which searches for a formal theory in which we can formalize all mathematical theories and prove all mathematical theorems, indicating that mathematics is essentially reducible to logic. However, G1 directly refutes this strong version of logicism: there is no theory in which we can prove all mathematical truths.

Fourthly, Gödel's incompleteness theorems have had a strong influence on the development of Hilbert's program. Two main goals of Hilbert's program are to show that **PA** is complete (all true arithmetic sentences are provable in **PA**) and to prove the consistency of arithmetic via finitistic methods. However, G1 directly refutes the first goal. It is usually thought that G2 also directly refutes the second goal of Hilbert's program, but different views on this have been reported [see Detlefsen (1979)]. We do not have a precise definition of the notion of a "finitistic method", with opinions about what a finitistic method is. If proving an arithmetic proposition A via a finitistic method means that A is provable in **PA**, then in this sense, we can say that G2 directly refutes the second goal of Hilbert's program if we express the consistency of **PA** as **Con**(**PA**). If we view transfinite induction with a countable length as a finitistic method, then Gentzen's result shows that we can prove the consistency of **PA** in some theory stronger than **PA** with respect to interpretation via finitistic methods. Thus, G2 does not directly refute Hilbert's program but indicates the limit of its applicability and shows that the full realization of Hilbert's program is impossible. Since Gödel's incompleteness theorems, Hilbert's program has been partially realized with great success in ordinal analysis and reverse mathematics. In the literature, there have been many discussions about the development of Hilbert's program since Gödel's incompleteness theorems and its influence on mathematical logic (especially proof theory) and the philosophy of mathematics [see Feferman (1988), Stephen (1988), Zach (2007)].

Fifthly, Gödel's incompleteness theorems are important negative results in the foundations of mathematics. Under their influence, many other important negative results in logic have been discovered, such as Tarski's theorem on the undefinability of truth, Church's undecidability theorem of first-order theories, Turing's undecidability theorem of the halting problem, and the independence of the continuum hypothesis, which are important and profound results in mathematical logic and the foundations of mathematics. These negative results reveal the limit of some key notions in logic such as provability, definability, computability, decidability, and independence.

3.2 Impact of Gödel's theorems on ordinary mathematics and theoretic computer science

Now we briefly discuss the impact of Gödel's incompleteness theorems on ordinary mathematics. A popular view is that most proofs of Gödel's incompleteness theorems use purely logical methods such as arithmetization and self-reference, which are unrelated to ordinary mathematics, and that the Gödel sentence is a purely logical construction without real mathematical content. Indeed, the Gödel sentence is not about properties of natural numbers but about properties of arithmetic theories themselves. In this sense, we can say that what Gödel's proof reveals is logical incompleteness. Despite Gödel's incompleteness theorems, one can still hope that **PA** is mathematically complete (i.e., all natural and mathematically interesting sentences about natural numbers are provable or refutable in **PA**).[16] A natural important question after Gödel is whether we find true sentences not provable in **PA** from ordinary mathematics with real mathematical content.

We refer to the incompleteness phenomenon revealed by independent sentences with real mathematical content as concrete incompleteness. The research program on concrete incompleteness is the search for independent sentences with real mathematical content from ordinary mathematics. Since Gödel, people have discovered in ordinary mathematics many arithmetic sentences independent of **PA** with real mathematical content whose constructions do not use arithmetization or the provability predicate. Refer to Beklemishev (2010), Bovykin (2006), and Cheng (2019) for examples of these sentences. An interesting and noteworthy fact is that many independent sentences from ordinary mathematics with real mathematical content are in fact equivalent to some metamathematical sentences via arithmetization [see

16 We say a sentence A is refutable in **PA** if ¬A is provable in **PA**.

Beklemishev (2010), Murawski (1999)]. This phenomenon indicates that the difference between mathematical and metamathematical statements is not as great as we might have expected, and there may be no clear dividing line between having and not having real mathematical content.

Research since Gödel has revealed the mathematical incompleteness of theories stronger than **PA** such as higher-order arithmetic and axiomatic set theory (**ZFC**). For example, Gödel and Cohen proved that the continuum hypothesis is independent of **ZFC**. In fact, statements independent of **ZFC** with real mathematical content have been found in many fields of mathematics such as analysis, algebra, topology, and mathematical logic.

Harvey Friedman is an international renowned expert in the foundations of mathematics since Gödel. He has made many important contributions to the research on concrete incompleteness. He writes: "the long range impact and significance of ongoing investigations in the foundations of mathematics is going to depend greatly on the extent to which the Incompleteness Phenomena touches normal concrete mathematics [see Friedman (2022, p. 7)]".

Friedman's work has extended the research on concrete incompleteness from first-order arithmetic to higher-order arithmetic and axiomatic set theory. For recent new advances in research on concrete incompleteness, refer to Friedman's monograph "Boolean Relation Theory and Incompleteness". Friedman (2022) proposed many independent statements of various sub-theories of **PA**, higher-order arithmetic, and **ZFC** from ordinary mathematics. For more research on concrete incompleteness, refer to Bovykin (2006), Cheng (2019), and Friedman (2022).

The research on concrete incompleteness is important, foundational, and profound, and reveals the incompleteness phenomenon and its influence on ordinary mathematics. Friedman's research program on concrete incompleteness shows that we can find independent statements of **ZFC** with real mathematical content in nearly any field of ordinary mathematics. Friedman's research program is important and promising: if it can be realized successfully, it will show that concrete incompleteness is ubiquitous in mathematics, which will be one of the most important discoveries in the foundations of mathematics since Gödel.

Now we briefly discuss the influence of Gödel's incompleteness theorems on theoretic computer science. An important research subject in theoretic computer science is the power and limitation of computation. Gödel's work is closely related to computability theory. Gödel's proof contains some fundamental ideas of computability theory such as arithmetization and recursive functions, which are two important tools in theoretic computer science. Recursive functions are one of the basic concepts and theoretical cornerstones of recursion theory (computability theory) developed since Gödel. The idea of arithmetization is to use natural numbers to encode some grammatical objects of a formal theory, such as terms, formulas, and proofs. Arithmetization is a key and useful technique in theoretical computer science and has played an important role in the development of recursion theory.

Completeness/incompleteness is also closely related to decidability/undecidability. We say a consistent theory T is decidable if the set of theorems of T is recursive; we say T is essentially undecidable if any recursively axiomatized consistent extension of T in $L(T)$ is undecidable. We can show that a theory T is essentially undecidable if and only if T is essentially incomplete.

In theoretic computer science, proving negative results is an important and unique tradition. A typical example is Turing's theorem on the undecidability of the halting problem. Turing's halting problem is closely related to Gödel's incompleteness theorems. We can prove G1 and G2 via the undecidability of the halting problem. Gödel proposed a precise mathematical definition of the general notion of a computable function in 1934. Then Church and Turing independently proposed different precise mathematical definitions of a computable function. Later, it was shown that all these different definitions of a computable function are equivalent. For more discussion about the influence of Gödel's incompleteness theorems on theoretic computer science, refer to Baaz (2014).

3.3 Impact of Gödel's theorems on philosophy

Different from other mathematical theorems, the impact of Gödel's incompleteness theorems is not confined to the community of mathematicians and logicians, and Gödel's theorems have also had a wide influence on philosophy.

The influence of Gödel's incompleteness theorems on philosophy is manifested in the fact that, since their publication, they have triggered extensive and enduring discussions of some philosophical problems related to them such as the nature of logic and mathematics, the nature of mind and machine, the difference between human intelligence and machine intelligence, the limit of machine intelligence, and the relationship among Gödel's theorems, logicism, Hilbert's program, and intuitionism. For the philosophical relevance of Gödel's incompleteness theorems, see Raatikainen (2005). For Gödel's philosophical thought and its influence on the discovery of his incompleteness theorems, see Wang (1990, 1997). There are many discussions in the literature about the philosophical significance of the incompleteness theorems and their applications in philosophy. However, the rationality of these applications in philosophy is controversial. For the correct use and misuse of the incompleteness theorems, see Franzen

(2005). In the following, we focus on two philosophical themes closely related to Gödel's incompleteness theorems: the anti-mechanist thesis and Gödel's disjunction thesis.

There have been many philosophical discussions based on Gödel's incompleteness theorems about whether the human mind can be mechanized [see Chalmers (1995), Cheng (2020a), Feferman (2009), Koellner (2018a, 2018b), Shapiro (1998)]. The mechanist thesis claims that the human mind can be mechanized and the anti-mechanist thesis claims that the human mind cannot be mechanized.[17]

Turing proposed a model of computation (the Turing machine) and gave a precise mathematical definition of the notion of "can be mechanized" via a Turing machine. Thus, the anti-mechanist thesis can be reformulated as "the mathematical outputs of the idealized human mind outstrip the mathematical outputs of any Turing machine". Here, the focus of our discussion is not the general philosophical question of whether the human mind can be mechanized, but the relationship between the incompleteness theorems and the anti-mechanist thesis.

A popular interpretation of G1 is that it implies the anti-mechanist thesis: the human mind cannot be mechanized. Many arguments in the literature support the anti-mechanist thesis based on the incompleteness theorems, among which the most famous are Lucas's argument and Penrose's argument [see Lucas (1996), Penrose (1989)]. Lucas's argument has been criticized widely in the literature.[18] Penrose (1994) proposed a new argument in support of the anti-mechanist thesis that is much more complex and dedicated than Lucas's argument. Penrose's new argument is one of the most sophisticated and promising anti-mechanist arguments up to now and has been extensively discussed and carefully analyzed in the literature [see Lindström (2006), Shapiro (2003), Koellner (2018b)]. The recent literature indicates that arguments in support of the anti-mechanist thesis (i.e., that G1 implies the anti-mechanist thesis) based on the incompleteness theorems come from some misinterpretations of Gödel's theorems [see Krajewski (2020), Cheng (2020a), Feferman (2009)]. Nowadays, most philosophers and logicians think that both Lucas's argument and variants of Penrose's arguments are not convincing. However, there are disagreements about the source of the mistakes of their arguments. Krajewski (2020) discussed in detail the history of arguments in support of the anti-mechanist thesis based on the incompleteness theorems, carefully analyzed

17 We will not consider the performance of actual human minds, with their limitations and defects; we only consider the idealized human mind and look at what it can do in principle (see Koellner 2018a, p. 338). In this paper, we will only consider the mathematical ability of the human mind: the mathematical truth the human mind can discover and prove.

18 For the history of the discussion of Lucas's argument, see Feferman (2009). For the influential criticism of Lucas's argument, see Benacerraf (1967).

the problems with Lucas's argument and variants of Penrose's argument, and concluded that G1 does not imply the anti-mechanist thesis. For more discussion about the relationship between the incompleteness theorems and the anti-mechanist thesis, refer to Koellner (2018a, 2018b).

Gödel did not argue that G1 implies that the human mind cannot be mechanized, even if he believed that the human mind in essence cannot be mechanized and is sufficiently powerful to capture all mathematical truths. For Gödel, mathematical proof is an essentially creative activity, and his incompleteness theorems indicate the creative power of human reason. Gödel believed that, compared with the Turing machine, the distinctiveness of the human mind is evident in its ability to come up with new axioms and develop new mathematical theories. Based on his rationalistic optimism, Gödel believed that the human mind is arithmetically omniscient and can capture all arithmetic truths.

However, Gödel admitted that he was unable to give a convincing argument that either "the human mind cannot be mechanized" or "there is no absolute undecidable proposition". Gödel thought that, from his incompleteness theorems, he could at most derive the following much weaker conclusion, known as Gödel's disjunctive thesis (GD).

[The first disjunct of GD]: The human mind cannot be mechanized.

[The second disjunct of GD]: There exists an absolutely undecidable statement.[19]

[Gödel's disjunctive thesis (GD)]: Either the first disjunct or the second disjunct holds.[20]

For Gödel, GD is a "mathematically established fact" of great philosophical interest that follows from his incompleteness theorems, and it is "entirely independent from the standpoint taken toward the foundation of mathematics" [Gödel (1951)]. For more discussion about GD, refer to Horsten and Welch (2016).

Now we discuss whether Gödel's incompleteness theorems imply GD. Our analysis of GD follows that of Koellner (2018a, 2018b). We first analyze a key notion in GD: absolute undecidability. Although we have a precise definition of provability relative to a formal theory, the notion of absolute undecidability is much vaguer, and we have no precise formal

19 In the sense that there are mathematical truths that cannot be proved by the idealized human mind.

20 The original version of GD was introduced by Gödel (1951, p. 310): "So the following disjunctive conclusion is inevitable: either mathematics is incompletable in this sense, that its evident axioms can never be comprised in a finite rule, that is to say, the human mind (even within the realm of pure mathematics) infinitely surpasses the powers of any finite machine, or else there exist absolutely unsolvable Diophantine problems of the type specified (where the case that both terms of the disjunction are true is not excluded, so that there are, strictly speaking, three alternatives)".

definition of it. The notion of absolute undecidability is not relative to some fixed formal theory and is essentially different from the notion of relative provability. There is little consensus on the properties of the notion of absolute undecidability.

Let $\langle M_n : n \in \omega \rangle$ be the list of all Turing machines. We call $Th(M_n)$ the output theory of Turing machine M_n (i.e., the set of all sentences M_n outputs). Let **K** be the set of all mathematical propositions the (ideal) human mind can know and **T** be the set of all true mathematical propositions.[21] We say a theory S is sound if $S \subseteq \mathbf{T}$. We assume that **K** is sound, and we equate absolute decidability with knowability of the (ideal) human mind. Under this assumption, **K** is the set of absolutely decidable mathematical propositions. Without specification, we assume that $Th(M_n)$ is an extension of **Q** such that we can apply both G1 and G2 to $Th(M_n)$. We say a proposition ϕ is absolutely undecidable if $\phi \notin \mathbf{K}$ and $\neg\phi \notin \mathbf{K}$. Now, we can reformulate GD as either $\neg\exists e(Th(M_e) = \mathbf{K})$ holds or $\exists\phi(\phi \in \mathbf{T} \wedge \phi \notin \mathbf{K} \wedge \neg\phi \notin \mathbf{K})$ holds.

Reinhardt (1986) proposed the theory $\mathbf{EA_T}$ and showed that if notions of relative provability, absolute undecidability, and truth satisfy some principles, then we can give a rigorous proof of GD, confirming Gödel's viewpoint that GD is a mathematical theorem [see Koellner (2018a, 2018b)]. The theory $\mathbf{EA_T}$ is a typed truth theory based on epistemic arithmetic, and its axioms include axioms of arithmetic, axioms about knowability, and axioms of typed truth theory.[22] Reinhardt (1986) proved that GD is provable in $\mathbf{EA_T}$.

On the basis of $\mathbf{EA_T}$, we can analyze the mechanist thesis in more detail and distinguish three forms of the mechanist thesis.

[the weak mechanist thesis (WMT)]: $\exists e(\mathbf{K} = Th(M_e))$;
[the strong mechanist thesis (SMT)]: $\mathbf{K}\exists e(\mathbf{K} = Th(M_e))$;
[the super-strong mechanist thesis (SSMT)]: $\exists e\mathbf{K}(\mathbf{K} = Th(M_e))$.

The mechanist thesis we discuss in this paper refers to WMT: there is some Turing machine M such that the human mind and M have the same mathematical outputs. The strong mechanist thesis claims that the human mind knows that there is some Turing machine M such that the human mind and M have the same mathematical outputs. The super-strong mechanist thesis claims that there is some Turing machine M such that the human mind knows that the human mind and M have the same mathematical outputs.

21 Gödel refers to **T** as objective mathematics and **K** as subjective mathematics.

22 For details of the theory $\mathbf{EA_T}$, refer to Koellner (2018a, 2018b).

Reinhardt (1986) proved that $\mathbf{EA_T}$ + SSMT is inconsistent. This indicates that in $\mathbf{EA_T}$, SSMT does not hold: there is no Turing machine *M* such that the human mind knows that the human mind and *M* have the same mathematical outputs. Reinhardt (1986) proved that $\mathbf{EA_T}$+ WMT is consistent, which indicates that the first disjunct of GD ($\neg\exists e(\mathbf{K} = Th(M_e))$) is not provable in $\mathbf{EA_T}$. Thus, from the viewpoint of $\mathbf{EA_T}$, it is possible that the human mind has the same mathematical output as some Turing machine. Carlson (2000) proved that $\mathbf{EA_T}$+ SMT is consistent, which indicates that, from the viewpoint of $\mathbf{EA_T}$ it is possible that the human mind knows that it is some Turing machine, but the human mind does not know which Turing machine it is. Thus, there is a major difference between $\mathbf{K}\exists e(\mathbf{K} = Th(M_e))$ and $\exists e\mathbf{K}(\mathbf{K} = Th(M_e))$.

Now we give a summary of the work on GD in Koellner (2018a, 2018b). Koellner thinks that to precisely discuss GD and prove that Gödel's incompleteness theorems imply GD, we should first properly formalize GD in some theory. On the basis of Reinhardt's work, Koellner proposed the theory DTK, a type-free truth theory based on epistemic arithmetic. Note that $\mathbf{EA_T}$ is a typed truth theory but DTK is a type-free truth theory. In DTK, we can formalize Penrose's new argument since it uses the notion of type-free truth [see Koellner (2018a, 2018b)]. Koellner formalized GD and its first and second disjuncts and proved that (1) GD is provable in DTK; (2) neither Lucas's argument nor Penrose's new argument holds in DTK; (3) both the first disjunct (the human mind cannot be mechanized) and the second disjunct (there is some absolute undecidable proposition) of GD are independent of DTK. Koellner concludes that it is possible that both "the human mind cannot be mechanized" and "there is some absolute undecidable proposition" are candidates of absolutely undecidable propositions [see Koellner (2018b)].

Koellner's work (2018a, 2018b) is the most comprehensive and precise analysis of GD in the literature as far as we know. Regardless of whether we agree with Koellner's analysis of GD, the significance of his work is that he successfully found a formal theory DTK in which we can formalize and prove GD, analyzed the reasons why Lucas's argument and Penrose's new argument fail, and proved the independence of the first and second disjuncts of GD from DTK. All these technical results strongly support Gödel's views on GD.

The word "deep" is often used to evaluate Gödel's incompleteness theorems, and views about what is a deep mathematical theorem differ among people. Even if it is agreed that some theorem is deep, different views may be held on the reason for this. We have no precise definition of mathematical depth, and we have no widely accepted general criteria to judge whether a given mathematical theorem is deep. Gödel's incompleteness theorems are widely viewed as deep mathematical theorems in logic. In Cheng (2022), Gödel's incompleteness theorems are used as a case study of mathematical depth, and the following two methodological issues are examined: the criteria for characterizing the depth of Gödel's incompleteness theorems and how to argue for the depth of Gödel's theorems on the basis of these criteria. On the basis of the recent research on incompleteness in the literature, three criteria are proposed in Cheng (2022) to characterize the depth of Gödel's incompleteness theorems (influence, fruitfulness, and unity), and the depth of Gödel's theorems is argued on the basis of these three criteria.

4 Limit of the Applicability of Gödel's Incompleteness Theorems

In this section, we examine the limit of the applicability of G1 and G2: under what conditions Gödel's theorems hold and do not hold.

4.1 Limit of the applicability of G1

We first give a brief discussion of the limit of the applicability of G1. Firstly, whether a theory is complete depends on the language of the theory. There are recursively axiomatized complete theories in the languages of $L(0, S)$, $L(0, S, <)$, and $L(0, S, <, +)$. For G1, including or interpreting enough information about arithmetic is crucial. For example, Euclidean geometry is a theory about points, lines, and surfaces rather than arithmetic. Tarski proved that the theory of Euclidean geometry is complete. If a theory only contains or interprets the arithmetic information of addition instead of multiplication, then the theory may be complete. For example, Presburger arithmetic is a complete theory in the language $L(0, S, <, +)$ [see Murawski (1999, Theorem 3.2.2)].

Secondly, G1 holds not only for all recursively axiomatized consistent extensions of **Q**, but also for some non-recursively axiomatized arithmetically definable consistent extensions of **Q**. However, not all arithmetically definable consistent extensions of **Q** are incomplete.[23]

The notion of interpretation provides us with a method to compare different theories in different languages [for the formal definition of interpretation, refer to Visser (2011)]. Informally, an interpretation of a theory *T* in a theory *S* is a mapping from formulas of *T* to formulas of *S* that maps all axioms of *T* to sentences provable in *S*. Given theories *S* and *T*, we say that *S* is weaker than *T* w.r.t. interpretation if *S* is interpretable in *T* but *T* is not interpretable

23 For examples of arithmetically definable consistent complete extensions of **Q**, refer to Salehi and Seraji (2017).

in S; we say that S is weaker than T w.r.t. the Turing degree if S is Turing reducible to T but T is not Turing reducible to S.

Thirdly, G1 can be generalized to many weaker theories than **PA** w.r.t. interpretation. Let T be a consistent recursively enumerable theory. We say G1 holds for T if for any recursively axiomatizable consistent theory S, if T is interpretable in S, then S is incomplete. Cheng (2020b) gives many examples of weaker theories than **PA** w.r.t. interpretation for which G1 holds.

Theory **R**, introduced by Tarski, Mostowski, and Robinson (1953), is another important base theory for investigating incompleteness and undecidability. Let **R** be the theory consisting of schemes **Ax1**–**Ax5** in the language {**0**, **S**, +, ×, ≤}, where ≤ is a primitive binary predicate symbol.

[Ax1]: $\overline{m}+\overline{n}=\overline{m+n}$;

[Ax2]: $\overline{m}\times\overline{n}=\overline{m\times n}$;

[Ax3]: $\overline{m}\neq\overline{n}$ if $m\neq n$;

[Ax4]: $\forall x(x\leq\overline{n}\rightarrow x=\overline{0}\vee\cdots\vee x=\overline{n})$;

[Ax5]: $\forall x(x\leq\overline{n}\vee\overline{n}\leq x)$.

Theory **R** is weaker than **Q** w.r.t. interpretation and contains all key properties of arithmetic for the proof of G1. It is well known that G1 holds for both **Q** and **R**. Although it is often thought that **R** is the weakest theory for which G1 holds, Cheng (2020b) showed that G1 holds for many theories weaker than **R** w.r.t. interpretation.

Fourthly, a natural question is whether there is a minimal theory for which G1 holds. The answer depends on our definition of minimality. If we define minimality as having the minimal number of axioms, then any finitely axiomatized essentially undecidable theory (e.g., Robinson Arithmetic **Q**) is a minimal theory for which G1 holds. For theories that are not finitely axiomatized, if we define minimality as having the minimal number of axiom schemes, then there is also a minimal theory for which G1 holds [see Cheng (2020b)]. When we talk about minimality, we should specify the degree structure involved. We say that S is a minimal recursively axiomatizable theory w.r.t. the Turing degree for which G1 holds if G1 holds for S and there is no recursively axiomatizable theory T such that G1 holds for T and T is weaker than S w.r.t. the Turing degree. Cheng (2020b) proved that there is no minimal recursively axiomatizable theory w.r.t. the Turing degree for which G1 holds.

4.2 Limit of the applicability of G2

In this section, we first discuss some generalizations of G2, then we examine the factors affecting whether G2 holds.

G2 also has many generalizations. Here, we only give some typical examples. Firstly, G2 also applies to some non-recursively axiomatized arithmetically definable extensions of **PA**. Secondly, G2 can also be generalized via the notion of interpretation. In fact, G2 applies to some weaker theories than **PA** w.r.t. interpretation. For example, a classic result from Pudlák is that there is no recursively axiomatized consistent theory T such that **Q** + **Con**(T) is interpretable in T. Thus, for any recursively axiomatized consistent theory T, if **Q** is interpretable in T, then G2 holds for T (**Con**(T) is not provable in T).

Thirdly, Löb's theorem is an important generalization of G2: assuming T is a recursively axiomatized consistent extension of **PA**, $\mathbf{Pr}_T(x)$ is a standard provability predicate, and ϕ is a formula in $L(T)$, then if $T \vdash \mathbf{Pr}_T(\ulcorner \phi \urcorner) \to \phi$, we have $T \vdash \phi$. As a corollary of Löb's theorem, G2 holds for T. Applying Löb's theorem to **PA**, if $\mathbf{PA} \nvdash \phi$, then $\mathbf{Pr}_{\mathbf{PA}}(\ulcorner \phi \urcorner) \to \phi$ is a true sentence in the standard model of arithmetic that is not provable in **PA**. Thus, from Löb's theorem, we have many arithmetic sentences independent of **PA**.

Löb's theorem reveals the relationship between Gödel's incompleteness theorems and provability logic. Historically, the origin of provability logic has been closely related to the incompleteness theorems. Provability logic is an important and useful tool for the study of incompleteness and the metamathematics of arithmetic. We know that the proof of G2 depends on properties of the provability predicate. Provability logic is the logic about properties of provability predicates, which provides us with an important viewpoint to understand incompleteness.

Both mathematically and philosophically, G2 is fundamentally different from G1. G1 concerns whether a theory is complete, i.e., whether the theory has an independent sentence. However, G2 concerns whether the consistency of the theory is provable in the theory itself. The consistency of a mathematical theory is a purely logical problem and is not the main concern of ordinary mathematics.

For the proof of G2, we need to properly formalize the notion "T is consistent". We say that G2 holds for a theory T if the consistency of T is not provable in T. However, this definition is vague: whether G2 holds for T depends on how we formalize the consistency of T. We call this the intensionality problem of G2. In the sense that we can construct independent sentences with real mathematical content from ordinary mathematics without the use of arithmetization and the provability predicate, we can say that G1 is extensional. However, G2 is intensional, and whether G2 holds for T depends on various factors as we will discuss. There are many discussions about the intensionality problem of G2 in the literature [see Halbach and Visser (2014a, 2014b), Visser (2011)]. On the basis of the current research on G2 in the literature, we give a summary of the factors determining whether G2 holds for a theory.

We say that a formula $\phi(x)$ is the representation formula of axioms of a theory T if for any natural number n, $\mathbf{PA} \vdash \phi(\overline{n})$ if and only if n is the code of some axiom of T. In the following, unless stated otherwise, we make the following assumptions:

(1) T is a recursively axiomatized consistent extension of $\mathbf{Q}$;
(2) the method of numbering used is Gödel numbering;
(3) the provability predicate used is standard;
(4) we use the canonical consistency statement $\mathbf{Con}(T) \triangleq \neg\mathbf{Pr}_T(\ulcorner \mathbf{0} \neq \mathbf{0} \urcorner)$ to express the consistency of T;[24]
(5) the representation formula of axioms of T is Σ_1^0.

We know that if a theory satisfies the above conditions, then G2 holds for it. On the basis of the research on G2 in the literature, we argue that whether G2 holds for T depends on the following factors:

(A) the choice of the base theory;
(B) the choice of the numbering;
(C) the choice of the provability predicate;
(D) the choice of the way of expressing consistency;
(E) the choice of the formula representing the set of axioms of the base theory.

We give some explanations of these factors. Firstly, these factors are not fully independent and their emphases differ. For example, given a fixed way of expressing consistency, using different provability predicates also leads to different ways of expressing consistency. However, (D) emphasizes the overall way of expressing consistency rather than the choice of specific provability predicates, whereas (C) emphasizes the choice of specific provability predicates.

Secondly, when we discuss how G2 depends on one factor, we always assume that all other factors are fixed according to the above assumptions, and we only discuss how G2 is

24 The canonical consistency statement uses Gödel numbering and a standard provability predicate.

affected by the factor we are discussing: if this factor does not satisfy the above assumption, whether and how G2 fails. Taking (C) for example, when we say that whether G2 holds for a theory T depends on the choice of the provability predicate, we mean that if we use a non-standard provability predicate to express the consistency of T, then G2 may fail for T, assuming that all other factors are fixed according to the above assumptions.

Now we give a brief summary of how G2 depends on the above five factors. For more detailed discussion of the intensionality problem of G2, refer to Cheng (2021). Firstly, whether G2 holds depends on the choice of the base theory. Proving G2 for an arithmetic theory needs more information of the arithmetic than that required to prove G1. If the base theory T does not contain enough information about the arithmetic, then G2 may fail for T: the consistency of T is provable in T [for examples, see Willard (2001, 2006)].

Secondly, whether G2 holds for a theory T depends on the choice of the numbering. Balthasar (2020) defined a class of acceptable numberings and proved that G2 holds for acceptable numberings assuming that other factors are fixed. Moreover, he pointed out if a numbering is not acceptable, then G2 may fail.

Thirdly, whether G2 holds for a theory T depends on the choice of the provability predicate used. We know that G2 holds for standard provability predicates. However, G2 may fail for non-standard provability predicates. An important non-standard provability predicate is the Rosser provability predicate, which was introduced by Rosser in his proof of G1. We can show that if we use the Rosser provability predicate in the canonical consistency statement, then G2 fails: if $\mathbf{Pr}_T^R(x)$ is a Rosser provability predicate, then the consistency statement defined as $\neg\mathbf{Pr}_T^R(\ulcorner \mathbf{0} \neq \mathbf{0} \urcorner)$ is provable in T.

Fourthly, whether G2 holds for a theory T depends on the way the consistency of T is expressed. We usually use an arithmetic sentence in $L(T)$ to express the consistency of T. However, there still are ways to express the consistency of T that are different from the canonical consistency statement such that the corresponding consistency statement is provable in T [see Taishi (2020)].

Finally, whether G2 holds for a theory T depends on the complexity of the representation formula of axioms of T. We know that if the representation formula of axioms of T is Σ_1^0, then the corresponding consistency statement is not provable in T. However, Feferman constructs a Π_1^0 representation formula of a theory T such that G2 fails for T: the corresponding consistency statement is provable in T [see Feferman (1960)].

In summary, both G1 and G2 apply to extensions of **PA** as well as some weak theories of arithmetic. These great variations in the generalizations of G1 and G2 reveal the explanatory power of the incompleteness theorems and their wide applicability. When we discuss the philosophical significance of G2, we should be aware of the scope and limit of the applicability of G2. The above discussion indicates that whether G2 holds depends on at least five factors. It may also depend on other factors such as the method of formalization and the logic used. To make the philosophical discussion of G2 meaningful, we need some absolute version of G2. An interesting question is whether we can find some more general conditions such that G2 is absolute with respect to these conditions: if a theory satisfies these conditions, then G2 holds for it.

5 Conclusion

The research on Gödel's incompleteness theorems has greatly updated and deepened our understanding of the incompleteness phenomenon. Many different proofs and generalizations of Gödel's theorems have been found via methods from various fields, and the scope and limit of the applicability of Gödel's incompleteness theorems have been discovered. The research on the incompleteness theorems has revealed the influence of Gödel's theorems on different fields, the fruitfulness of Gödel's theorems, and the relationship among the incompleteness theorems, ordinary mathematics, undecidability, logical paradox, provability logic, and so forth. Gödel himself may not have expected the following: his incompleteness theorems have such an important and profound impact on the foundations of mathematics, philosophy, theoretic computer science, and ordinary mathematics; his incompleteness theorems have many distinct proofs and generalizations; and his second incompleteness theorem depends on many factors.

On the impact of Gödel's incompleteness theorems, Feferman said "their relevance to mathematical logic (and its offspring in the theory of computation) is paramount; further, their philosophical relevance is significant, but in just what way is far from settled; and finally, their mathematical relevance outside of logic is very much unsubstantiated but is the object of ongoing, tantalizing efforts" [see Feferman (2006, p. 434)].

As Feferman summarized, the influence of the incompleteness theorems on the foundations of mathematics and theoretic computer science is obvious and huge and will be lasting and far-reaching. The philosophical significance of Gödel's incompleteness theorems and their application in philosophy are popular topics of philosophical discussions, but they are also full of controversy; a consensus has not yet been reached on the philosophical significance of the incompleteness theorems and their application in philosophy. The influence of the incompleteness theorems on mathematics is currently mainly reflected in the research on concrete incompleteness, with many independent statements having been found in ordinary mathematics. The importance and influence of the incompleteness theorems on mathematics depend largely on the extent to which Friedman's research program on concrete incompleteness has been realized. If in the future we can reveal that incompleteness is everywhere in almost every field of ordinary mathematics, the incompleteness theorems will have a greater impact on ordinary mathematics.

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