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Research article

Modal Logics for the Poison Game: Axiomatization and Undecidability

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Abstract:
In the tradition of memory logic, two comparatively weak systems, poison modal logic (PML) and poison sabotage logic (PSL), were studied in existing literature to capture the so-called poison game, which originally served as a paradigm to reason about graph-theoretical notions and was recently shown to have important applications in the theory of abstract argumentation. In this work, we continue to explore the technical aspect of the two logics and complete the existing results by providing our solutions to the questions identified in the literature (Grossi and Rey, 2019a; 2019b; Blando *et al*., 2020). Precisely, we show that (i) neither of them can be embedded in fixed-variable fragment of first-order logic, on the basis of the existing findings for PML and PSL, (ii) PSL has an undecidable satisfiability problem, and (iii) motivated by the existing axiomatization results for the standard memory logic (Areces *et al*., 2012), we axiomatize the two logics in a broader setting with enrichments from hybrid logic. As we shall see, in line with the fact that PSL is strictly weaker than PML, the calculus for the hybrid PSL developed will also be a 'fragment' of that for the hybrid PML.

Keywords:
Memory logic, Hybrid logic, Graph game logic, Credulously admissible set, Axiomatization, Undecidability

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1 Poison Games and Memory Logic

The poison game is introduced by Duchet and Meyniel (1993) to reason about some graph-theoretical notions (e.g., the so-called '*semi-kernels*') in graph theory. It is a zero-sum perfect information game played by two players, *Mover* and *Poisoner*, on a directed graph (W, R). The game is started by Mover that needs to choose a node s from W, and in each round afterwards, Poisoner chooses a successor t of the state chosen by Mover in the previous stage (this means state t is *poisoned*), and then Mover selects a non-poisoned state from among the successors of t. Poisoner wins if and only if Mover cannot make a legal movement. In addition to its applications in graph theory, as indicated in the works of Grossi and Rey (2019a; 2019b), the framework also has natural applications in the abstract argumentation theory, in that it can determine the existence of *credulously admissible sets* in an argumentation framework. To reason about the game, many efforts have been made to design suitable modal logics fitting with the game, referring to the standard memory logic whose language is equipped with tools to check whether or not a given state has already been memorized and to memorize a state (Areces *et al*., 2012; Areces *et al*., 2009; Areces *et al*., 2011; Areces *et al*., 2008). In this paper, we focus on two existing modal logics for poison games, poison modal logic (PML) and poison sabotage logic (PSL), and study their logical properties that provide solutions to questions that are left in the literature. However, before we move to the details, let us first introduce the existing results for the logics in the literature and identify the questions that we will study.

The framework PML is independently proposed in the works of Blando *et al*. (2020) and Grossi and Rey (2019a) (and its extension (Grossi and Rey, 2019b)). As shown in the same works, there are natural notions of first-order translation and bisimulation for the logic, which together lead to a van Benthem style characterization theorem identifying the counterpart of the logic in first-order logic (FOL) (Grossi and Rey, 2019b). With the help of these, the logic is shown to be strictly weaker than the standard memory logic. Also, PML can be embedded into the hybrid logic with nominals and binders (Grossi and Rey, 2019b). So far, many properties of the computational behavior of the logic have also been studied: for instance, the logic lacks the finite model property, its satisfiability problem is undecidable, and its model-checking problem is PSPACE-complete.

In addition to PML, Blando *et al*. (2020) also developed the proposal PSL for the purpose of designing logical tools with lower computational complexity. Indeed, the logic is strictly weaker than PML. Many results for PML can be easily adapted to fit with PSL, including the notions of first-order translation and bisimulation. Moreover, as the case for PML, the model-checking problem for PSL is shown to be PSPACE-complete.

Although many properties of the two logics have been explored, there are still several questions involving their technical properties that were identified in the works of Grossi and Rey (2019a; 2019b); Blando *et al*. (2020). In this work, we will provide solutions to the following:

- Can PML and PSL be embedded into fixed-variable fragments of FOL or not? (Grossi and Rey, 2019b)[1]
- Can we have desired Hilbert-style proof systems for PML and PSL? (Grossi and Rey, 2019b)
- Is the satisfiability problem for PSL decidable? (Blando *et al*., 2020)

As we shall see, with the findings of Blando *et al*. (2020), we can have a quick, but negative, answer to the first question. Also, to provide a solution to the second questions, we extend the two logics with devices from hybrid logic, which is motivated by the techniques developed for the standard memory logic (Areces *et al*., 2012). Finally, by encoding the well-known undecidable $\mathbb{N} \times \mathbb{N}$-tiling problem, we will offer a negative answer to the last question.

Outline. In Section 2, we recall the designs of PML and PSL, identify their 'minimal base logics,' extend them with nominals and @-operators from hybrid logic, and show that, among others, the answer to the first question is negative, referring to existing findings in the literature. Then, in Section 3, we prove that the satisfiability problem for PSL is undecidable. Next, complete Hilbert-style proof systems for the hybrid augments of PML and PSL are developed in Section 4, and finally, Section 5 concludes with several further directions.

2 Logics for the Poison Game and Their Hybrid Extensions

In this part, we first introduce the designs of PML and PSL. Then, we define the corresponding minimal logics, show the definitions of their hybrid extensions and study some of their basic properties, whereby we will have an answer to the question whether or not the two logics can be embedded in fixed-variable fragments of FOL. We start with the language of PML.

1 This question and the second one below are asked only for PML in the work of Grossi and Rey (2019b), but they also make sense for PSL.

Definition 1. *Let* Prop *be a countable set of propositional letters. The* language $\mathcal{L}_{\mathsf{M}}$ for PML *is given in the following manner:*

$$\mathcal{L}_{\mathsf{M}} \ni \varphi ::= p \mid \mathfrak{p} \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Diamond\varphi \mid \langle \mathfrak{p} \rangle\varphi$$

where $p \in$ Prop, $\mathfrak{p}$ *is a propositional constant, and* $\langle \mathfrak{p} \rangle$ *is a modality. Also, we use* $\Box$ *and* $[\mathfrak{p}]$ *for the dual operators of* $\Diamond$ *and* $\langle \mathfrak{p} \rangle$, *respectively*.

The readings of basic modal formulas are as usual. The constant $\mathfrak{p}$ indicates *the current state is memorized*, and $\langle \mathfrak{p} \rangle\varphi$ expresses that *the current state has a successor u s.t. after we memorize u, φ is true at u*. Given φ, ψ, χ, we use $\varphi[\chi/\psi]$ for the formula obtained by replacing all occurrences of χ in φ with ψ.

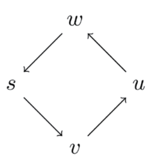
The *models for* PML are tuples $\mathfrak{M} = (W, R, V, S)$, where $W \neq \emptyset$, $R \subseteq W \times W$ is a binary relation on W, V : Prop $\rightarrow \mathcal{P}(W)$ is a valuation function,[2] and $S \subseteq W$ is a *memory set* consisting of the states that have been memorized. For each $w \in W$, we define $R(w) \coloneqq \{u : \langle w, u\rangle \in R\}$, denoting the set of successors of the state w. The semantics for PML is given by the following:

Definition 2. *Let* $\mathfrak{M} = (W, R, V, S)$ *be a model and* $w \in W$. *The* truth of formulas $\varphi \in \mathcal{L}_{\mathsf{M}}$ at w in $\mathfrak{M}$, *written* $\mathfrak{M}, w \vDash \varphi$, *is recursively defined in the following manner:*

$$\begin{array}{lll}
\mathfrak{M}, w \vDash p & \textit{iff} & w \in V(p) \\
\mathfrak{M}, w \vDash \mathfrak{p} & \textit{iff} & w \in S \\
\mathfrak{M}, w \vDash \neg\varphi & \textit{iff} & \mathfrak{M}, w \nvDash \varphi \\
\mathfrak{M}, w \vDash \varphi \wedge \psi & \textit{iff} & \mathfrak{M}, w \vDash \varphi \textit{ and } \mathfrak{M}, w \vDash \psi \\
\mathfrak{M}, w \vDash \Diamond\varphi & \textit{iff} & \mathfrak{M}, u \vDash \varphi \textit{ for some } u \in R(w) \\
\mathfrak{M}, w \vDash \langle \mathfrak{p} \rangle\varphi & \textit{iff} & \mathfrak{M}|_{\langle v + \mathfrak{p} \rangle}, v \vDash \varphi \textit{ for some } v \in R(w)
\end{array}$$

where $\mathfrak{M}|_{\langle v + \mathfrak{p} \rangle} \coloneqq (W, R, V, S \cup \{v\})$.

Example 1. To see how the device works, let us consider a model $\mathfrak{M} = (W, R, V, \emptyset)$, where $V(p) = \emptyset$ for any $p \in$ Prop. The formula $\langle \mathfrak{p} \rangle\langle \mathfrak{p} \rangle\langle \mathfrak{p} \rangle\Diamond\Diamond\mathfrak{p}$ is true at w in $\mathfrak{M}$, since $(W, R, V, \{s, u, v\}), u \vDash \Diamond\Diamond\mathfrak{p}$.



2 For any set A, we use $\mathcal{P}(A)$ for its *power set*.

The notions of *satisfiability*, *validity* and *logical consequence* are defined as the usual. PML refers to the validities with respect to the class of models whose memory sets start with the empty set, which is in line with the fact that the set of states that have been poisoned in a game always begins with $\varnothing$. Also, we call the logic defined w.r.t. the class of all models 'the minimal poison modal logic' abbreviated as MPML. All validities of MPML are valid in PML, but validities of the latter need not be valid in the former: for a witness, $\neg \mathfrak{p}$ is a validity of PML, but not of MPML.

Moreover, the *language* $\mathcal{L}_{\mathsf{S}}$ *for* PSL can be obtained by replacing $\Diamond\varphi$ in $\mathcal{L}_{\mathsf{M}}$ with a new modality $\langle t\rangle\varphi$, reading *the current state has a successor u s.t. u is not memorized and φ is true at u*. The truth condition is as follows:

$$\mathfrak{M}, w \vDash \langle t\rangle\varphi \textit{ iff } \mathfrak{M}, v \vDash \varphi \text{ for some } v \in R(w) \setminus S$$

Moreover, we will use $[t]$ for the dual of $\langle t\rangle$.

Reconsider the model $\mathfrak{M}$ defined in Example 1. When we substitute $\Diamond$ with $\langle t\rangle$ in $\langle \mathfrak{p} \rangle\langle \mathfrak{p} \rangle\langle \mathfrak{p} \rangle\Diamond\Diamond \mathfrak{p}$, the resulting formula $\langle \mathfrak{p} \rangle\langle \mathfrak{p} \rangle\langle \mathfrak{p} \rangle\langle t\rangle\langle t\rangle \mathfrak{p}$ would be false at w in $\mathfrak{M}$: by the truth condition for $\langle t\rangle$, formula $\langle t\rangle \mathfrak{p}$ is always false.

Again, PSL refers to the validities w.r.t. the class of models whose memory sets begin with the empty set, and we use MPSL for the corresponding minimal logic, i.e., the validities w.r.t. the class of all models. It is easy to see that PSL and MPSL are fragments of PML and MPML respectively, since we can define $\langle t\rangle\varphi$ as $\Diamond(\neg \mathfrak{p} \wedge \varphi)$. In terms of the expressiveness on models, PSL is strictly weaker than PML, which is also strictly weaker than the standard memory logic (Blando *et al*., 2020).

Also, we can show the following with the existing results developed for the two logics:

Proposition 1. *Neither* PML *or* PSL *can be embedded in fixed-variable fragments of* FOL.

Proof. The model-checking problems for both PSL and PML are PSPACE-complete (Blando *et al*., 2020) and the two logics can be embedded into FOL with functions that have a polynominal size increase (Blando *et al*., 2020; Grossi and Rey, 2019b). On the other hand, Vardi (1995) showed that the model-checking problem for any fragment of FOL with a fixed number of variables is in P. These together show that neither PML nor PSL can be translated into fixed-variable fragments of FOL. □

In what follows, we will also work with the hybrid extension, HPML, of PML. Precisely, the *language* $\mathcal{L}_{\mathsf{HM}}$ *for* HPML is given in the following manner:

$$\mathcal{L}_{\mathsf{HM}} \ni \varphi ::= i \mid p \mid \mathfrak{p} \mid \neg\varphi \mid \varphi \wedge \varphi \mid @_i\varphi \mid \Diamond\varphi \mid \langle \mathfrak{p} \rangle\varphi$$

where $i \in \mathsf{Nom}$ and $\mathsf{Nom} = \{i, j, k, ...\}$ is a set of nominals such that $\mathsf{Nom} \cap \mathsf{Prop} = \emptyset$.

Models $\mathfrak{M} = (W, R, V, S)$ for HPML are the same as before, except that the valuations V now are functions from $\mathsf{Prop} \cup \mathsf{Nom}$ to $\mathcal{P}(W)$. As usual, nominals i are interpreted as singletons (i.e., $\{w\}$). In what follows, if there is no danger of confusion, we use $\overline{i}$ to denote the node named i. The truth conditions for i and $@_i\varphi$ are as follows:

$$\mathfrak{M}, w \vDash i \textit{ iff } w = \overline{i}$$
$$\mathfrak{M}, w \vDash @_i\varphi \textit{ iff } \mathfrak{M}, \overline{i} \vDash \varphi$$

As before, we can also obtain a fragment of HPML by replacing $\Diamond\varphi$ in $\mathcal{L}_{\mathsf{HM}}$ with $\langle t\rangle\varphi$, and we write $\mathcal{L}_{\mathsf{HS}}$ for the resulting language and write HPSL for the logic. It can be shown that HPSL is still strictly weaker than HPML.[3] Also, we use HMPML and HMPSL for the hybrid extensions of MPML and MPSL, respectively. It is easy to see that HMPSL is also a fragment of HMPML.

With our design, we can show a property of locality for HMPML (and HMPSL). For any formula $\varphi \in \mathcal{L}_{\mathsf{HM}}$, we use $\mathsf{Prop}(\varphi)$ for the set of all propositional variables occurring in φ and use $\mathsf{Nom}(\varphi)$ for the set of nominals occurring in the formula. Now we can show the following:

Proposition 2. *Let $\varphi \in \mathcal{L}_{\mathsf{HM}}$, $\mathfrak{M}_1 = (W, R, V_1, S)$ and $\mathfrak{M}_2 = (W, R, V_2, S)$ be two models such that for any $x \in \mathsf{Prop}(\varphi) \cup \mathsf{Nom}(\varphi)$, $V_1(x) = V_2(x)$. For any $w \in W$, it holds that*

$$\mathfrak{M}_1, w \vDash \varphi \textit{ iff } \mathfrak{M}_2, w \vDash \varphi.$$

As a consequence, for all formulas of $\mathcal{L}_{\mathsf{HS}}$, we also have the same result.

Proof. It goes by induction on formulas. Hybrid formulas can be proved in the usual way, and the case for $\mathfrak{p}$ is trivial. Thus, we just consider the case that $\varphi \equiv \langle \mathfrak{p} \rangle\psi$. Details are as follows:

$$\begin{aligned} \mathfrak{M}_1, w \vDash \langle \mathfrak{p} \rangle\psi &\text{ iff } (W, R, V_1, S \cup \{v\}), v \vDash \psi \text{ for some } v \in R(w) \\ &\text{ iff } (W, R, V_2, S \cup \{v\}), v \vDash \psi \text{ for some } v \in R(w) \end{aligned}$$

3 To give a precise proof for this, we need the notions of bisimulation for the logics, but we leave the details to another occasion. For the notions of bisimulation for PML and PSL, we refer to Blando *et al*. (2020), and for the notion of bisimulation for the hybrid logic, we refer to ten Cate (2005).

iff $\mathfrak{M}_2, w \vDash \langle \mathfrak{p} \rangle \psi$

The second equivalence holds by induction hypothesis. □

3 Undecidability for PSL

In this section, we aim to study the computational behavior of PSL, and we will prove that the satisfiability problem for PSL is undecidable. To achieve our goal, we will show that the undecidable $\mathbb{N} \times \mathbb{N}$-tiling problem (Berger, 1966) can be encoded with the satisfiability problem for PSL.

Let $\mathcal{T} = \{t_1, ..., t_n\} \subseteq \mathsf{Prop}$ be a finite set of tile types. We use *right*(t_k) for the set of tile types that can be placed to the right of tile type t_k, and *top*(t_k) for the set of tile types that can be placed to the top of tile type t_k. For a non-empty and finite set T of formulas, $\bigvee T \coloneqq \bigvee_{\varphi \in T} \varphi$, and $oneT \coloneqq \bigvee_{\varphi \in T} (\varphi \wedge \bigwedge_{\psi \in T \setminus \{\varphi\}} \neg \psi)$, meaning that there is exactly one formula in T that is true.

In the construction below, we will make use of the techniques of 'spy point' (Blackburn and Seligman, 1995) (all states that are reachable from a spy point in n-steps can also be reached in one step), and we will use a propositional letter 's' for such states. Moreover, we use propositional letters '0,' '1,' and '2' for tiles, and the letters 'u' and 'r' to denote, respectively, upward movements and movements to the right. For any $x \in \{u, r\}$, we will use $\Diamond_x \varphi$ for $\langle t \rangle (x \wedge \langle t \rangle \varphi)$ and $\Box_x \varphi$ for its dual. We use R_{up} and R_{right} for the relations described by $\Diamond_u$ and $\Diamond_r$, respectively. To be precise,

$$R_{up} \coloneqq \{\langle w_1, w_2 \rangle : \mathfrak{M}, w_{1(2)} \vDash \bigvee \{0, 1, 2\} \textit{ and there is a u-state } w_3 \textit{ s.t. } \langle w_1, w_3 \rangle, \langle w_3, w_2 \rangle \in R\}$$
$$R_{right} \coloneqq \{\langle w_1, w_2 \rangle : \mathfrak{M}, w_{1(2)} \vDash \bigvee \{0, 1, 2\} \textit{ and there is an r-state } w_3 \textit{ s.t. } \langle w_1, w_3 \rangle, \langle w_3, w_2 \rangle \in R\}$$

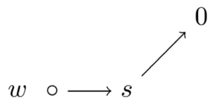
Intuitively, $\langle w_1, w_2 \rangle \in R_{up}$ means tile w_2 is above w_1, and $\langle w_1, w_2 \rangle \in R_{right}$ means w_2 is to the right of w_1. Now we define a desired formula $\varphi_{\mathcal{T}}$ that is the conjunction of (T1)–(T19) constructed below. For ease of understanding, we will gradually draw the key features of a potential model, which are required to make the formula true.

(T1) $\neg s \wedge \bigwedge_{i \in \{0,1,2,u,r\}} \neg i \wedge \langle t \rangle s$

This shows that the current state (e.g., w) is not 0, 1, 2, u, r or s, but can reach at least one s-state.

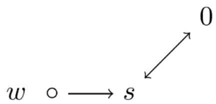
(T2) $[t](s \wedge \bigwedge_{i \in \{0,1,2,u,r\}} \neg i \wedge \langle t \rangle 0 \wedge [t](one\{0, 1, 2, u, r\} \wedge \neg s \wedge \bigwedge_{i \in \{0,1,2,u,r\}} (i \rightarrow \neg \langle t \rangle i))$

All successors of *w* are *s*-points, and each of these *s*-states satisfies the following: (a) it is not 0, 1, 2, *u* or *r*; (b) it can reach at least one 0-state; (c) each successor of such an *s*-point is irreflexive and satisfies exactly one of 0, 1, 2, *u*, *r* but not *s*.



(T3) [ᵽ]$[t]\neg\langle t \rangle s \wedge [t][t]\langle t \rangle s$

By this formula, fixing an *s*-successor, the successors of the *s*-state can also see an *s*-state, which is exactly the fixed *s*-state itself. As we shall see, one of the *s*-states will serve as a spy point.



(T4) $[t][t][t](\neg s \rightarrow \langle t \rangle s) \wedge$ [ᵽ]$[t][t](\neg s \rightarrow \neg\langle t \rangle s)$

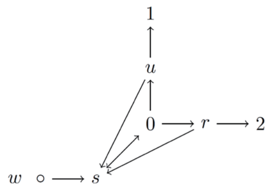
By fixing an *s*-successor, all the ¬*s*-states accessible from the *s*-state in two steps can reach the *s*-state in one step (but cannot reach other different *s*-states).

In the following, we define the relations R_{up} and R_{right} for tiles {0, 1, 2}, which represent 'moving up' and 'moving to the right' respectively. For each tile $i \in \{0, 1, 2\}$, let $\mathsf{u}(i) = (i + 1)$ mod 3, $\mathsf{r}(i) = (i + 2)$ mod 3. Note that we have the following:

$$\{i, \mathsf{u}(i), \mathsf{u}(\mathsf{u}(i))\} = \{i, \mathsf{r}(i), \mathsf{r}(\mathsf{r}(i))\} = \{0, 1, 2\} \text{ and } \mathsf{u}(\mathsf{r}(i)) = \mathsf{r}(\mathsf{u}(i)) = i.$$

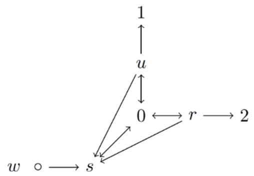
(T5) $\bigwedge_{i \in \{0,1,2\}} [t][t](i \rightarrow \Diamond_u \mathsf{u}(i)) \wedge \bigwedge_{i \in \{0,1,2\}} [t][t](i \rightarrow \Diamond_r \mathsf{r}(i))$

This means that every *i*-tile, which is accessible from an *s*-state in one step, has at least one $\mathsf{u}(i)$-tile above it and at least one $\mathsf{r}(i)$-tile to its right. Combining this and (T4), we have the following:



(T6) $\bigwedge_{i\in\{0,1,2\}} [t][\mathfrak{p}](i \to [t](\neg s \to \neg\langle t\rangle i)) \wedge \bigwedge_{i\in\{0,1,2\}} [t][t](i \to [t](\neg s \to \langle t\rangle i))$

For any *i*-tile *a* accessible from an *s*-state, the relation from *a* to its *u*-successors and *r*-successors is symmetric, and those *u*-successors and *r*-successors cannot see other *i*-tiles except *a*.



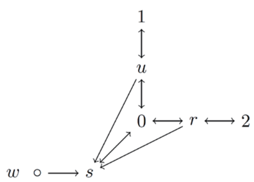
(T7) $\bigwedge_{i\in\{0,1,2\}} [t][t][\mathfrak{p}]((u \wedge \langle t\rangle \mathsf{u}(i) \wedge \langle t\rangle i) \to [t](\neg \mathrm{s} \to \neg\langle t\rangle(u \wedge \langle t\rangle \mathsf{u}(i) \wedge \langle t\rangle i))) \wedge$

$\bigwedge_{i\in\{0,1,2\}} [t][t][\mathrm{t}]((u \wedge \langle t\rangle \mathsf{u}(i) \wedge \langle t\rangle i) \to [t](\neg \mathrm{s} \to \langle t\rangle(u \wedge \langle t\rangle \mathsf{u}(i) \wedge \langle t\rangle i)))$

(T8) $\bigwedge_{i\in\{0,1,2\}} [t][t][\mathfrak{p}]((r \wedge \langle t\rangle \mathsf{r}(i) \wedge \langle t\rangle i) \to [t](\neg \mathrm{s} \to \neg\langle t\rangle(r \wedge \langle t\rangle \mathsf{r}(i) \wedge \langle t\rangle \mathrm{i}))) \wedge$

$\bigwedge_{i\in\{0,1,2\}} [t][t][t]((r \wedge \langle t\rangle \mathsf{r}(i) \wedge \langle t\rangle i) \to [t](\neg \mathrm{s} \to \langle t\rangle(r \wedge \langle t\rangle \mathsf{r}(i) \wedge \langle t\rangle i)))$

By (T7), given a *u*-state *a* that is accessible from an *s*-state in two steps and can reach both *i* and $\mathsf{u}(i)$, any $\neg s$-state that can be reached from *a* can see a *u*-state, which is exactly *a* itself. (T8) is analogous to (T7) and deals with *r*-states. Now the previous picture becomes the following:



(T9) $\bigwedge_{i\in\{0,1,2\}} [t][t](i \to [t](u \to (\langle t\rangle \mathsf{u}(i) \leftrightarrow \neg\langle t\rangle \mathsf{u}(\mathsf{u}(i))))) \wedge$

$$\bigwedge_{i\in\{0,1,2\}} [t][t](i \rightarrow [t](r \rightarrow (\langle t\rangle \mathsf{r}(i) \leftrightarrow \neg\langle t\rangle \mathsf{r}(\mathsf{r}(i)))))$$

The formula puts further restrictions to the links between u-states, r-states, and tiles i. For instance, fixing an s-state and an i-tile a that can be reached, for any u-state $b \in R(a)$, b can reach a $\mathsf{u}(i)$-tile or a $\mathsf{u}(\mathsf{u}(i))$-tile, but not both.

(T10) $$\bigwedge_{i\in\{0,1,2\}} [t][\text{ᵽ}](i \rightarrow [t](u \rightarrow [\text{ᵽ}](\mathsf{u}(i) \rightarrow [\text{ᵽ}]((u \wedge \neg\langle t\rangle i \wedge \neg\langle t\rangle \mathsf{u}(i) \wedge \neg\langle t\rangle \mathsf{u}(\mathsf{u}(i)))$$

$$\rightarrow [t](s \rightarrow \neg\langle t\rangle(u \wedge \neg\langle t\rangle i \wedge \neg\langle t\rangle \mathsf{u}(i) \wedge \neg\langle t\rangle \mathsf{u}(\mathsf{u}(i)))))))) \wedge$$

$$\bigwedge_{i\in\{0,1,2\}} [t][\text{ᵽ}](i \rightarrow [t](u \rightarrow [\text{ᵽ}](\mathsf{u}(i) \rightarrow [t]((u \wedge \neg\langle t\rangle i \wedge \neg\langle t\rangle \mathsf{u}(i) \wedge \neg\langle t\rangle \mathsf{u}(\mathsf{u}(i)))$$

$$\rightarrow [t](s \rightarrow \langle t\rangle(u \wedge \neg\langle t\rangle i \wedge \neg\langle t\rangle \mathsf{u}(i) \wedge \neg\langle t\rangle \mathsf{u}(\mathsf{u}(i))))))))$$

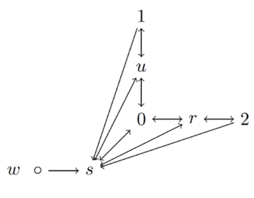
(T11) $$\bigwedge_{i\in\{0,1,2\}} [t][\text{ᵽ}](i \rightarrow [t](r \rightarrow [\text{ᵽ}](\mathsf{r}(i) \rightarrow [\text{ᵽ}]((r \wedge \neg\langle t\rangle i \wedge \neg\langle t\rangle \mathsf{r}(i) \wedge \neg\langle t\rangle \mathsf{r}(\mathsf{r}(i)))$$

$$\rightarrow [t](s \rightarrow \neg\langle t\rangle(r \wedge \neg\langle t\rangle i \wedge \neg\langle t\rangle \mathsf{r}(i) \wedge \neg\langle t\rangle \mathsf{r}(\mathsf{r}(i)))))))) \wedge$$

$$\bigwedge_{i\in\{0,1,2\}} [t][\text{ᵽ}](i \rightarrow [t](r \rightarrow [\text{ᵽ}](\mathsf{r}(i) \rightarrow [t]((r \wedge \neg\langle t\rangle i \wedge \neg\langle t\rangle \mathsf{r}(i) \wedge \neg\langle t\rangle \mathsf{r}(\mathsf{r}(i)))$$

$$\rightarrow [t](s \rightarrow \langle t\rangle(r \wedge \neg\langle t\rangle i \wedge \neg\langle t\rangle \mathsf{r}(i) \wedge \neg\langle t\rangle \mathsf{r}(\mathsf{r}(i))))))))$$

The two formulas are analogous: the former is about u-states and the latter is about r-states. By (T10), given an s-state a and an i-tile $b \in R(a)$, any u-successor of a can also be reached from that s-state a. On the basis of this formula and (T4), we have the following:



(T12) $$\bigwedge_{i\in\{0,1,2\}} [t][\text{ᵽ}](i \rightarrow \Box_u(\dagger \rightarrow \Box_u\neg i)) \wedge \bigwedge_{i\in\{0,1,2\}} [t][t](i \rightarrow \Box_u(\dagger \rightarrow \Diamond_u i)), (\dagger \in \{\mathsf{u}(i), \mathsf{u}(\mathsf{u}(i))\})$$

(T13) $$\bigwedge_{i\in\{0,1,2\}} [t][\text{ᵽ}](i \rightarrow \Box_r(\dagger \rightarrow \Box_r\neg i)) \wedge \bigwedge_{i\in\{0,1,2\}} [t][t](i \rightarrow \Box_r(\dagger \rightarrow \Diamond_r i)), (\dagger \in \{\mathsf{r}(i), \mathsf{r}(\mathsf{r}(i))\})$$

Formulas (T12) and (T13) ensure that R_{up}, R_{right}, and their inverse relations are functional.

$$\text{(T14)} \bigwedge_{i\in\{0,1,2\}} [t][t](i \to [t](r \to [\text{ꝑ}](\mathsf{r}(i) \to [t](r \to [t](i \to [t](u \to [t](\mathsf{u}(i)\wedge\Diamond_u(i\wedge\neg\Diamond_r\mathsf{r}(i)))$$

$$\to [t](s \to\langle t\rangle(\mathsf{u}(i) \wedge \Diamond_u(i \wedge \neg\Diamond_r\mathsf{r}(i)))))))))) \wedge$$

$$\bigwedge_{i\in\{0,1,2\}} [t][t](i \to [t](r \to [\text{ꝑ}](\mathsf{r}(i) \to [t](r \to [t](i \to [t](u \to [\text{ꝑ}](\mathsf{u}(i)\wedge\Diamond_u(i\wedge\neg\Diamond_r\mathsf{r}(i)))$$

$$\to [t](s \to\neg\langle t\rangle(\mathsf{u}(i) \wedge \Diamond_u(i \wedge \neg\Diamond_r\mathsf{r}(i))))))))))$$

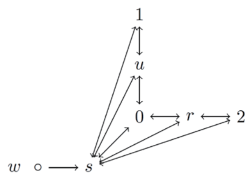
$$\text{(T15)} \bigwedge_{i\in\{0,1,2\}} [t][t](i \to [t](u \to [\text{ꝑ}](\mathsf{u}(i) \to [t](u \to [t](i \to [t](r \to [t](\mathsf{r}(i) \wedge \Diamond_r(i \wedge \neg\Diamond_u\mathsf{u}(i)))$$

$$\to [t](s \to \langle t\rangle(\mathsf{r}(i) \wedge \Diamond_r(i \wedge \neg\Diamond_u\mathsf{u}(i)))))))))) \wedge$$

$$\bigwedge_{i\in\{0,1,2\}} [t][t](i \to [t](u \to [\text{ꝑ}](\mathsf{u}(i) \to [t](u \to [t](i \to [t](r \to [\text{ꝑ}](\mathsf{r}(i) \wedge \Diamond_r(i \wedge \neg\Diamond_u\mathsf{u}(i)))$$

$$\to [t](s \to \neg\langle t\rangle(\mathsf{r}(i) \wedge \Diamond_r(i \wedge \neg\Diamond_u\mathsf{u}(i))))))))))$$

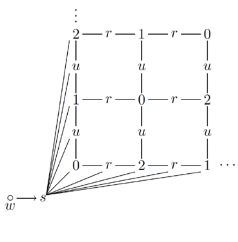
By these two formulas, in the picture, the *s*-state is seen to reach the 1-state, guaranteed by (T14), and can also reach the 2-state, guaranteed by (T15).



$$\text{(T16)} \bigwedge_{i\in\{0,1,2\}} [t][t](\mathsf{u}(i) \to \Box_u(i \to \Box_r(\mathsf{r}(i) \to \Diamond_u(i \wedge \Diamond_r\mathsf{u}(i))))) \wedge$$

$$\bigwedge_{i\in\{0,1,2\}} [t][\text{ꝑ}](\mathsf{u}(i) \to \Box_u(i \to \Box_r(\mathsf{r}(i) \to \Box_u(i \to\Box_r \neg\mathsf{u}(i)))))$$

Formula (T16) guarantees the *confluence property*. Then, based on (T5), we have the following, where we use the undirected links to mean the links with two directions:



(T17) $\bigwedge_{i\in\{0,1,2\}} [t][t](i \to one\mathcal{T})$

(T18) $\bigwedge_{i\in\{0,1,2\},1\leq k\leq n} [t][t](i \wedge t_k \to \Box_u(\mathsf{u}(i) \to \bigvee top(t_k)))$

(T19) $\bigwedge_{i\in\{0,1,2\},1\leq k\leq n} [t][t](i \wedge t_k \to \Box_r(\mathsf{r}(i) \to \bigvee right(t_k)))$

Lemma 1. *If $\mathcal{T}$ tiles $\mathbb{N} \times \mathbb{N}$, then $\varphi_{\mathcal{T}}$ is satisfiable.*

Proof. Let $g : \mathbb{N} \times \mathbb{N} \to \mathcal{T}$ be a tiling function. We define a model $\mathfrak{M}_g = (W_g, R_g, V_g, \emptyset)$ as follows:

- $W_g = W_0 \cup \{w, w^*\}$ and $W_0 = \{\langle n, m\rangle \in \mathbb{N} \times \mathbb{N} : n \times m \text{ is even}\}$
- $R_g = R_u \cup R_r \cup \{\langle w, w^*\rangle\} \cup (\{w^*\} \times W_0) \cup (W_0 \times \{w^*\})$ where
  - $R_u = \{\langle\langle 2k, l\rangle, \langle 2k, l+1\rangle\rangle : k, l \in \mathbb{N}\} \cup \{\langle\langle 2k, l+1\rangle, \langle 2k, l\rangle\rangle : k, l \in \mathbb{N}\}$
  - $R_r = \{\langle\langle k, 2l\rangle, \langle \mathrm{k}+1, 2l\rangle\rangle : k, l \in \mathbb{N}\} \cup \{\langle\langle k+1, 2l\rangle, \langle k, 2l\rangle\rangle : k, l \in \mathbb{N}\}$
- V_g is defined as
  - $V_g(s) = \{w^*\}$
  - $V_g(u) = \{\langle 2k, 2l+1\rangle : k, l \in \mathbb{N}\}$
  - $V_g(r) = \{\langle 2k+1, 2l\rangle : k, l \in \mathbb{N}\}$
  - $V_g(0) = \{\langle 6k, 6l\rangle : k, l \in \mathbb{N}\} \cup \{\langle 6k+2, 6l+2\rangle : k, l \in \mathbb{N}\} \cup \{\langle 6k+4, 6l+4\rangle : k, l \in \mathbb{N}\}$
  - $V_g(1) = \{\langle 6k, 6l+2\rangle : k, l \in \mathbb{N}\} \cup \{\langle 6k+2, 6l+4\rangle : k, l \in \mathbb{N}\} \cup \{\langle 6k+4, 6l\rangle : k, l \in \mathbb{N}\}$
  - $V_g(2) = \{\langle 6k, 6l+4\rangle : k, l \in \mathbb{N}\} \cup \{\langle 6k+2, 6l\rangle : k, l \in \mathbb{N}\} \cup \{\langle 6k+4, 6l+2\rangle : k, l \in \mathbb{N}\}$
  - $V_g(t_k) = \{\langle 2k, 2l\rangle : k, l \in \mathbb{N}, g(k, l) = t_k\}$ for each $1 \leq k \leq n$
  - $V_g(p) = \emptyset$ for each $p \in \mathsf{Prop} \setminus (\mathcal{T} \cup \{s, u, r, 0, 1, 2\})$

For an illustration of a piece of information about the model, consider the final picture directed for our formulas constructed: the s-state is w^*, the 0-point in the lower left is $\langle 0, 0\rangle$, the u-point above $\langle 0, 0\rangle$ is $\langle 0, 1\rangle$, the r-state to the left of the center 0-point is $\langle 1, 2\rangle$, and so on. Now, $\mathfrak{M}_g, w \vDash \varphi_{\mathcal{T}}$. □

Lemma 2. *If $\varphi_{\mathcal{T}}$ is satisfiable, then $\mathcal{T}$ tiles $\mathbb{N} \times \mathbb{N}$.*

Proof. Let $\mathfrak{M} = (W, R, V, \emptyset)$ be a model and $w \in W$ such that $\mathfrak{M}, w \vDash \varphi_{\mathcal{T}}$. We define a tiling function $g : \mathbb{N} \times \mathbb{N} \to \mathcal{T}$. In what follows, for any φ, we use $[\![\varphi]\!] \subseteq W$ for the set of states where φ is true.

By (T2) and (T17), we have $[\![\bigvee \mathcal{T}]\!] \neq \emptyset$. Then by (T5) and (T12), for each $w_1 \in [\![\bigvee \mathcal{T}]\!]$, there is a state $w_2 \in [\![\bigvee \mathcal{T}]\!]$ s.t. $w_1 R w_u R w_2$ where $w_u \in V(u)$, $w_1 \in V(i)$ and $w_2 \in V(\mathsf{u}(i))$ for some

$i \in \{0, 1, 2\}$, and we denote w_2 as $up(w_1)$ and note that $up : [\![\bigvee \mathcal{T}]\!] \to [\![\bigvee \mathcal{T}]\!]$ is a function. Similarly, by (T5) and (T13), for each $w_1 \in [\![\bigvee \mathcal{T}]\!]$, there is a $w_2 \in [\![\bigvee \mathcal{T}]\!]$ s.t. $w_1 R w_r R w_2$ where $w_r \in V(r)$, $w_1 \in V(i)$ and $w_2 \in V(\mathsf{r}(i))$ for some $i \in \{0, 1, 2\}$, and we denote w_2 as $right(w_1)$ and note that $right : [\![\bigvee \mathcal{T}]\!] \to [\![\bigvee \mathcal{T}]\!]$ is a function. Let $w_0 \in [\![\bigvee \mathcal{T}]\!]$. We define $g_1 : \mathbb{N} \times \mathbb{N} \to [\![\bigvee \mathcal{T}]\!]$ as follows:

- $g_1(\langle 0, 0\rangle) = w_0$
- $g_1(\langle n, m + 1\rangle) = up(g_1(\langle n, m\rangle))$
- $g_1(\langle n + 1, m\rangle) = right(g_1(\langle n, m\rangle))$ for $m, n \in \mathbb{N}$.

By (T16), we have

$$g_1(\langle n + 1, m + 1\rangle) = up(right(g_1(\langle n, m\rangle))) = right(up(g_1(\langle n, m\rangle))).$$

Thus g_1 is well defined. Then we define $g_2 : [\![\bigvee \mathcal{T}]\!] \to \mathcal{T}$ as follows.

$$\text{For each } 1 \leq k \leq n,\ g_2(v) = t_k \text{ iff } v \in V(t_k).$$

Let $g = g_2 \circ g_1$. Then by (T17)–(T19), g is a tiling function. □

Theorem 1. PSL *is undecidable.*

Proof. It follows from Lemmas 1 and 2 directly. □

In the undecidability proof, the modality $\langle \mathfrak{p} \rangle\varphi$ is crucial in defining the tiling problem, and dropping it from the language would lead us to a decidable fragment,[4] but the fragment obtained by removing the constant $\mathfrak{p}$ from PSL is still undecidable; the constant is not used in the construction at all. However, in contrast, both the fragments of PML (and MPML) without $\mathfrak{p}$ and without $\langle \mathfrak{p} \rangle$ are decidable; the former fragment is just the usual polymodal logic with two modalities (for the same relation) and the latter is exactly the basic modal logic (again, the constant $\mathfrak{p}$ in such a setting can still be treated as an ordinary propositional atom).

4 Recall that models for PSL have $\emptyset$ as their initial memory sets, and so lacking the tool $\langle \mathfrak{p} \rangle\varphi$ that can extend memory sets makes $\mathfrak{p}$ is always false, which then indicates that $\langle t\rangle\varphi$ behaves in the same way as the modality $\Diamond\varphi$. More generally, w.r.t. an arbitrary model of MPSL whose memory set might not be empty, $\langle t\rangle\varphi$ can still be defined as $\Diamond(\neg \mathfrak{p} \wedge \varphi)$ of the basic modal language, in which $\mathfrak{p}$ is treated as an ordinary propositional atom.

4 Axiomatization

Although the complexity of the logics is high, this part will show that there are still complete Hilbert-style proof systems for HPML and HPSL. For this, we will first show desired calculi for the minimal proposals HMPML and HMPSL. Our techniques are motivated by Areces *et al*. (2012) in which complete proof systems were developed for several hybrid extensions of the standard memory logic, but many modifications are needed to fit with the new designs. As we shall see, in line with the fact that HMPSL is a fragment of HMPML, the calculus for the former provided in this section is also a 'fragment' of the calculus for the latter.[5] Moreover, both the proof systems enjoy the strong completeness properties.

4.1 Calculus for HMPML

We start by considering HMPML. A desired calculus **HMPML** for the logic is given in Table 1. It extends axioms and rules for the hybrid logic with axioms and rules for $[\,\mathfrak{p}\,]$ (**II**) and the interaction axioms between $[\,\mathfrak{p}\,]$ and $\Box$ (**III**). Derivations in **HMPML** are defined as usual. For each formula φ, we write $\vdash_{\mathbf{HMPML}} \varphi$ if there is a derivation of φ in **HMPML**. When there is no confusion, we also write $\vdash\varphi$ for simplicity.

The rule ($\mathsf{Paste}_{\langle\,\mathfrak{p}\,\rangle}$) is an analogy of ($\mathsf{Paste}_{\Diamond}$), but we need to take care of the augment of the memory set. Formula (Memory) means that *if (a) i can reach j and (b) j is φ after we add j to the memory set, then i has a successor u such that after adding u to the memory set, φ is true at u.* Moreover, ($\mathsf{Com}\langle\,\mathfrak{p}\,\rangle\Diamond$) states *if (a) i can reach j after we add j to the memory set and (b) j is φ (before the enlargement of the memory set), then i can reach j in the initial model.* Finally, ($\mathsf{Com}\Diamond\langle\,\mathfrak{p}\,\rangle$) intuitively means that the enlargement of the memory set does not affect the accessibility relation.

In the proof system, the form $\varphi[\,\mathfrak{p}\,/\,\mathfrak{p}\ \vee j]$ is involved, which is intended to capture the case of extending the memory set with j. Note that $(\langle\,\mathfrak{p}\,\rangle\varphi)[\,\mathfrak{p}\,/\,\mathfrak{p}\ \vee j]$ is $\langle\,\mathfrak{p}\,\rangle(\varphi[\,\mathfrak{p}\,/\,\mathfrak{p}\ \vee j])$. This can be made precise as follows:

Lemma 3. *Let $\mathfrak{M} = (W, R, V, S)$ be a model and $w \in W$ such that $V(i) = w$. Then for any $v \in W$ and any φ, the following holds*:

5 We believe that the calculus for HMPML is also a 'fragment' of the calculus for the standard memory logic in the work of Areces *et al*. (2012), but we will not show this to keep this article compact.

Table 1. A proof system **HMPML** for HMPML.

I: Axioms and rules for hybrid logic	
(Tau)	Propositional tautologies
($K_\Box$)	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
($K_@$)	$@_i(\varphi \to \psi) \to (@_i\varphi \to @_i\psi)$
($Dual_\Box$)	$\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi$
(SelfDual)	$\neg @_i\varphi \leftrightarrow @_i\neg\varphi$, where $i \in$ Nom.
(Ref)	$@_i i$, where $i \in$ Nom.
(Intro)	$i \wedge \varphi \to @_i\varphi$, where $i \in$ Nom.
($Back_\Box$)	$@_i\varphi \to \Box @_i\varphi$, where $i \in$ Nom.
(Agree)	$@_j @_i\varphi \to @_i\varphi$, where $i, j \in$ Nom.
(MP)	From $\varphi \to \psi$ and φ, infer ψ.
($Nec_\Box$)	From φ, infer $\Box\varphi$.
($Nec_@$)	From φ, infer $@_i\varphi$, where $i \in$ Nom.
(Name)	From $i \to \varphi$, infer φ, where $i \in$ Nom is new to φ.
($Paste_\Diamond$)	From $@_i\Diamond j \wedge @_j\varphi \to \psi$, infer $@_i\Diamond\varphi \to \psi$, where j is new to φ, ψ, i.
II: Axioms and rules for $[\mathfrak{p}]$	
($K_{[\mathfrak{p}]}$)	$[\mathfrak{p}](\varphi \to \psi) \to ([\mathfrak{p}]\varphi \to [\mathfrak{p}]\psi)$
($Dual_{[\mathfrak{p}]}$)	$\langle\mathfrak{p}\rangle\varphi \leftrightarrow \neg[\mathfrak{p}]\neg\varphi$
(Memory)	$@_i\langle\mathfrak{p}\rangle j \wedge @_j\varphi[\mathfrak{p}/\mathfrak{p} \vee j] \to @_i\langle\mathfrak{p}\rangle\varphi$
($Paste_{\langle\mathfrak{p}\rangle}$)	From $@_i\langle\mathfrak{p}\rangle j \wedge @_j\varphi[\mathfrak{p}/\mathfrak{p} \vee j] \to \psi$, infer $@_i\langle\mathfrak{p}\rangle\varphi \to \psi$, where j is new to φ, ψ, i.
III: Interaction axioms	
($Com\langle\mathfrak{p}\rangle\Diamond$)	$@_i\langle\mathfrak{p}\rangle j \wedge @_j\varphi \to @_i\Diamond\varphi$
($Com\Diamond\langle\mathfrak{p}\rangle$)	$@_i\Diamond j \to @_i\langle\mathfrak{p}\rangle j$

$$(W, R, V, S \cup \{w\}), v \vDash \varphi \textit{ iff } (W, R, V, S), v \vDash \varphi[\mathfrak{p}/\mathfrak{p} \vee i]$$

Proof. We prove this lemma by induction on φ. The cases for $x \in$ Prop $\cup$ Nom and Boolean connectives $\neg$, $\wedge$ are easy, and we show the proof for other cases.

(1) $\varphi \equiv \mathfrak{p}$. Then, $\varphi[\mathfrak{p}/\mathfrak{p} \vee i] \equiv \mathfrak{p} \vee i$. For the direction from left to right, we assume that $(W, R, V, S \cup \{w\}), v \vDash \mathfrak{p}$. Then, it holds immediately by the semantics that $v \in S \cup \{w\}$. If $v = w$, then $\mathfrak{M}, v \vDash i$ and so $\mathfrak{M}, v \vDash \mathfrak{p} \vee i$; if $v \neq w$, then $\mathfrak{M}, v \vDash \mathfrak{p}$. Thus, we always have $\mathfrak{M}, v \vDash \mathfrak{p} \vee i$.

For the direction from right to left, suppose that $(W, R, V, S), v \vDash \mathfrak{p} \vee i$. Then, there are also two different cases. Firstly, if $(W, R, V, S), v \vDash \mathfrak{p}$, then $v \in S$ and so $(W, R, V, S \cup \{w\}), v \vDash \mathfrak{p}$. Secondly, if $(W, R, V, S), v \vDash i$, then $v = w$ and so $(W, R, V, S \cup \{w\}), v \vDash \mathfrak{p}$.

(2) $\varphi \equiv @_j\psi$. Then, it follows that

$$\begin{aligned}(W, R, V, S \cup \{w\}), v \vDash @_j\psi &\text{ iff } (W, R, V, S \cup \{w\}), \overline{j} \vDash \psi \\ &\text{ iff } (W, R, V, S), \overline{j} \vDash \psi[\mathfrak{p} / \mathfrak{p} \vee i] \\ &\text{ iff } (W, R, V, S), v \vDash @_j\psi[\mathfrak{p} / \mathfrak{p} \vee i]\end{aligned}$$

The second equivalence holds by the induction hypothesis.

(3) $\varphi \equiv \Diamond\psi$. The following sequence of equivalences holds:

$$\begin{aligned}(W, R, V, S \cup \{w\}), v \vDash \Diamond\psi &\text{ iff } (W, R, V, S \cup \{w\}), u \vDash \psi \text{ for some } u \in R(v) \\ &\text{ iff } (W, R, V, S), u \vDash \psi[\mathfrak{p} / \mathfrak{p} \vee i] \\ &\text{ iff } (W, R, V, S), v \vDash \Diamond\psi[\mathfrak{p} / \mathfrak{p} \vee i]\end{aligned}$$

The second equivalence holds by the induction hypothesis.

(4) $\varphi \equiv \langle \mathfrak{p} \rangle\psi$. Then we have the following:

$$\begin{aligned}(W, R, V, S \cup \{w\}), v \vDash \langle \mathfrak{p} \rangle\psi &\text{ iff } (W, R, V, S \cup \{w\} \cup \{u\}), u \vDash \psi \text{ for some } u \in R(v) \\ &\text{ iff } (W, R, V, S \cup \{u\}), u \vDash \psi[\mathfrak{p} / \mathfrak{p} \vee i] \\ &\text{ iff } (W, R, V, S), v \vDash \langle \mathfrak{p} \rangle\psi[\mathfrak{p} / \mathfrak{p} \vee i].\end{aligned}$$

The second one holds by the induction hypothesis. This completes the proof. □

Theorem 2. *For all formulas* φ, $\vdash_{\mathbf{HMPML}} \varphi$ *implies that* φ *is a validity of* HMPML.

Proof. We merely consider the axiom (Memory) and the rule (Paste$_{\langle \mathfrak{p} \rangle}$), as other cases are easy to show or similar to the cases of hybrid logic. Let $\mathfrak{M} = (W, R, V, S)$ be a model and $w \in W$.

(1) We first show the validity of (Memory). Suppose that $\mathfrak{M}, w \vDash @_i\langle \mathfrak{p} \rangle j \wedge @_j\varphi[\mathfrak{p} / \mathfrak{p} \vee j]$. Then, $\overline{j} \in \mathrm{R}(\overline{i})$ and $\mathfrak{M}, \overline{j} \vDash \varphi[\mathfrak{p} / \mathfrak{p} \vee j]$. By Lemma 3, $(W, R, V, S \cup \{\overline{j}\}), \overline{j} \vDash \varphi$. Hence, $\mathfrak{M}, \overline{i} \vDash \langle \mathfrak{p} \rangle\varphi$. Therefore, $\mathfrak{M}, w \vDash @_i\langle \mathfrak{p} \rangle\varphi$.

(2) We now move to the case for (Paste$_{\langle \mathfrak{p} \rangle}$). Suppose for reductio that $\mathfrak{M}, w \vDash @_i\langle \mathfrak{p} \rangle\varphi$ and $\mathfrak{M}, w \nvDash \psi$. Hence, $\mathfrak{M}, \overline{i} \vDash \langle \mathfrak{p} \rangle\varphi$ and so there exists $v \in R(\overline{i})$ s.t. $(W, R, V, S \cup \{v\}), v \vDash \varphi$. Let $\mathfrak{M}^* = (W, R, V^*, S)$, where

$$V^*(x) = \begin{cases} V(x), & \text{if } x \in \mathsf{Prop} \cup (\mathsf{Nom} \setminus \{j\}); \\ \{v\}, & x \equiv j. \end{cases}$$

Since j is new to φ, ψ and i, by Proposition 2, $(W, R, V^*, S \cup \{v\}), v \vDash \varphi$ and $\mathfrak{M}^*, w \nvDash \psi$. By Lemma 3, $\mathfrak{M}^*, v \vDash \varphi[\mathfrak{p} / \mathfrak{p} \vee j]$, which then gives us $\mathfrak{M}^*, w \vDash @_j\varphi[\mathfrak{p} / \mathfrak{p} \vee j]$. Note that $\overline{j}$

$\in R(\bar{i})$. Then, it holds that $\mathfrak{M}^*, w \vDash @_i\langle \mathfrak{p} \rangle j$. Hence, $\mathfrak{M}^*, w \vDash @_i\langle \mathfrak{p} \rangle j \wedge @_j\varphi[\mathfrak{p} / \mathfrak{p} \vee j]$ and $\mathfrak{M}^*, w \nvDash \psi$. Therefore, $(@_i\langle \mathfrak{p} \rangle j \wedge @_j\varphi[\mathfrak{p} / \mathfrak{p} \vee j]) \rightarrow \psi$ is not a validity of HMPML. □

4.2 Completeness for HMPML

Now we are going to show the completeness of the calculus **HMPML**. To do so, let us first introduce the following auxiliary notions.

Definition 3. Let $\Gamma \subseteq \mathcal{L}_{\mathsf{HM}}$ *be a set of formulas.*

- Γ *is* **HMPML**-consistent (*or consistent, if the proof system is clear from the context*) if $\nvdash_{\mathbf{HMPML}} \varphi_1 \wedge \cdots \wedge \varphi_n \rightarrow \bot$ *for any* $(\varphi_i)_{1\leq i\leq n} \subseteq \Gamma$*, and it is* **HMPML**-inconsistent *(or* inconsistent, *if the proof system is clear from the context) if it is not consistent.*
- Γ *is* maximal **HMPML**-consistent *if* Γ *is consistent and* Δ *is inconsistent for any* $\Gamma \subsetneq \Delta \subseteq \mathcal{L}$.
- Γ *is* named *if* $i \in \Gamma$ *for some nominal* $i \in \mathsf{Nom}$.
- Γ *is* pasted *if for each* $@_i\Diamond\varphi \in \Gamma$*, there is some nominal* j *such that* $@_i\Diamond j \wedge @_j\varphi \in \Gamma$.
- Γ *is* $\mathfrak{p}$-pasted *if for each* $@_i\langle \mathfrak{p} \rangle\varphi \in \Gamma$*, there is some nominal* j *such that* $@_i\langle \mathfrak{p} \rangle j \wedge @_j\varphi[\mathfrak{p} / \mathfrak{p} \vee j] \in \Gamma$.

A set Γ is called an *maximal* **HMPML**-*consistent set* (**HMPML**-MCS) if it is maximal **HMPML**-consistent. Let Nom' be a countable set of nominals disjoint from both Nom and Prop. Let $\mathsf{Nom}^+ = \mathsf{Nom} \cup \mathsf{Nom}'$ and let $\mathcal{L}^+_{\mathsf{HM}}$ be the same as $\mathcal{L}_{\mathsf{HM}}$ except that the nominals of $\mathcal{L}^+_{\mathsf{HM}}$ are those of $\mathsf{Nom+}$.

Lemma 4. *Any* **HMPML**-*consistent set of the language* $\mathcal{L}_{\mathsf{HM}}$ *can be extended to a named, pasted and* $\mathfrak{p}$-*pasted* **HMPML**-*MCS of the language* $\mathcal{L}^+_{\mathsf{HM}}$.

Proof. Let Γ be an **HMPML**-consistent set, $j_0 \in \mathsf{Nom}'$ and $\Gamma_0 := \Gamma \cup \{j_0\}$. With the rule (Name), one can check that Γ_0 is consistent. Let $(\varphi_n)_{n\in\mathbb{N}}$ be an enumeration of all formulas of $\mathcal{L}^+_{\mathsf{HM}}$. Then for each $k \in \mathbb{N}$, we define the set Γ_{k+1} as follows.

- If $\Gamma_k \cup \{\varphi_k\}$ is not consistent, then $\Gamma_{k+1} := \Gamma_k$.
- If $\Gamma_k \cup \{\varphi_k\}$ is consistent, then
  - $\Gamma_{k+1} = \Gamma_k \cup \{\varphi_k\} \cup \{@_i\Diamond j \wedge @_j\psi\}$ if φ_k is of the form $@_i\Diamond\psi$, where $j \in \mathsf{Nom+}$ is the first new nominal with respect to Γ_k and φ_k.
  - $\Gamma_{k+1} = \Gamma_k \cup \{\varphi_k\} \cup \{@_i\langle \mathfrak{p} \rangle j \wedge @_j\psi[\mathfrak{p} / \mathfrak{p} \vee j]\}$ if φ_k is of the form $@_i\langle \mathfrak{p} \rangle\psi$, where $j \in \mathsf{Nom+}$ is the first new nominal with respect to Γ_k and φ_k.

• $\Gamma_{k+1} = \Gamma_k \cup \{\varphi_k\}$ if φ_k is not of the form $@_i\Diamond\psi$ or $@_i\langle \mathfrak{p} \rangle\psi$.

Let $\Gamma^* = \bigcup_{n\in\mathbb{N}}\Gamma_n$. By the rules ($\mathsf{Paste}_{\Diamond}$) and ($\mathsf{Paste}_{\langle \mathfrak{p} \rangle}$), Γ_k is consistent for all $k \in \mathbb{N}$. The set Γ^* is an **HMPML**-MCS that extends Γ. Moreover, by the construction, Γ^* is named, pasted and $\mathfrak{p}$-pasted, as needed. □

Here are a few facts from hybrid logic.

Proposition 3 *Let Γ be an* **HMPML**-*MCS of* $\mathcal{L}^+_{\mathsf{HM}}$. *For each $i \in$ Nom+, let $\Delta_i \coloneqq \{\varphi : @_i\varphi \in \Gamma\}$.*

(1) *For all $i \in$ Nom+, Δ_i is an* **HMPML**-*MCS and $i \in \Delta_i$.*

(2) *$\Gamma = \Delta_i$ whenever $i \in \Gamma$.*

(3) *For all $i, j \in$ Nom+, $i \in \Delta_j$ implies $\Delta_i = \Delta_j$.*

Proof. These are standard results for the hybrid logic (Blackburn *et al.*, 2001, p. 439, Lemma 7.24). □

Now we are going to show the strong completeness of **HMPML**. By Lemma 4, any **HMPML**-consistent set Δ can be extended to a named, pasted and $\mathfrak{p}$-pasted **HMPML**-MCS $\Gamma \subseteq \mathcal{L}^+_{\mathsf{HM}}$. So, to achieve the goal, it suffices to show that Γ is satisfiable.

Definition 4. *Let $\Gamma \subseteq \mathcal{L}^+_{\mathsf{HM}}$ be an* **HMPML**-*MCS that is named, pasted and $\mathfrak{p}$-pasted. The* canonical model induced by *Γ is the tuple $\mathfrak{M}^\Gamma = \langle W^\Gamma, R^\Gamma, V^\Gamma, S^\Gamma\rangle$, where*

- $W^\Gamma \coloneqq \{\Delta_i : i \in \mathsf{Nom+}\}$,
- $R^\Gamma\Delta_i\Delta_j$ *iff* $@_i\langle \mathfrak{p} \rangle j \in \Gamma$,
- $V^\Gamma(x) \coloneqq \{\Delta_i \in W^\Gamma : x \in \Delta_i\}$, *for all $x \in$ Prop $\cup$ Nom+, and*
- $S^\Gamma \coloneqq \{\Delta_i \in W^\Gamma : \mathfrak{p} \in \Delta_i\}$.

Since Γ is named, there is a nominal $j \in \Gamma$. From item (2) of Proposition 3 it follows that $\Gamma = \Delta_j$, which indicates that $\Gamma \in W^\Gamma$. Now, a key step towards the proof of completeness is to show a truth lemma, for which we define the following.

Definition 5. *We define* the complexity of $\mathcal{L}_{\mathsf{HM}}$-formulas *as*

$$C(\varphi) = 4 \times (|\mathfrak{p}| + 1) \times (|\langle \mathfrak{p} \rangle| + 1) + |\neg| + |\Diamond| + |@| + |\wedge|$$

where $|\circ|$ for $\circ \in \{\mathfrak{p}, \langle \mathfrak{p} \rangle, \neg, \Diamond, @, \wedge\}$ is the number of occurrences of $\circ$ in φ.

It is important to emphasis that $|\mathfrak{p}|$ does not include the number of occurrences of $\mathfrak{p}$ in the operator $\langle \mathfrak{p} \rangle$. The above notion can ensure that $C(\langle \mathfrak{p} \rangle\varphi) > C(\varphi[\mathfrak{p}/\mathfrak{p} \vee i])$.[6] Now we proceed to show the following *Truth Lemma*.

Lemma 5. *Let* $\varphi \in \mathcal{L}^{+}_{\mathsf{HM}}$ *and* $\Delta_i \in W^{\Gamma}$. *Then*

$$\mathfrak{M}^{\Gamma}, \Delta_i \vDash \varphi \textit{ iff } \varphi \in \Delta_i.$$

Proof. We prove this lemma by induction on $C(\varphi)$. The cases for $\mathsf{Prop} \cup \{\mathfrak{p}\} \cup \mathsf{Nom+}$ and Boolean connectives are trivial. Also, the case for $\varphi \equiv @_j\psi$ holds by the same reason as that of the hybrid logic. We now just consider other cases.

(1) $\varphi \equiv \Diamond\psi$. For the direction from left to right, assume that $\mathfrak{M}^{\Gamma}, \Delta_i \vDash \Diamond\psi$. Then, there is a set $\Delta_j \in R^{\Gamma}(\Delta_i)$ s.t. $\mathfrak{M}^{\Gamma}, \Delta_j \vDash \psi$. By induction hypothesis, we have $\psi \in \Delta_j$ and so $@_j\psi \in \Gamma$. Moreover, by Definition 4, $@_i\langle \mathfrak{p} \rangle j \in \Gamma$. Therefore, $@_i\langle \mathfrak{p} \rangle j \wedge @_j\psi \in \Gamma$. Then, it follows from the axiom ($\mathsf{Com}\langle \mathfrak{p} \rangle\Diamond$) and the rule ($\mathsf{MP}$) that $@_i\Diamond\psi \in \Gamma$. Thus, $\Diamond\psi \in \Delta_i$.

For the direction from right to left, we suppose that $\Diamond\psi \in \Delta_i$. Immediately, $@_i\Diamond\psi \in \Gamma$. Since Γ is pasted, there exists a nominal $j \in \mathsf{Nom+}$ s.t. $@_i\Diamond j \wedge @_j\psi \in \Gamma$. Then by ($\mathsf{Com}\Diamond\langle \mathfrak{p} \rangle$) and ($\mathsf{MP}$), it holds that $@_i\langle \mathfrak{p} \rangle j \in \Gamma$. Then $\Delta_j \in R^{\Gamma}(\Delta_i)$. Note that $\varphi \in \Delta_j$. By induction hypothesis, $\mathfrak{M}^{\Gamma}, \Delta_j \vDash \psi$. Hence, $\mathfrak{M}^{\Gamma}, \Delta_i \vDash \Diamond\psi$.

(2) $\varphi \equiv \langle \mathfrak{p} \rangle\psi$. For the direction from left to right, we assume that $\mathfrak{M}^{\Gamma}, \Delta_i \vDash \langle \mathfrak{p} \rangle\psi$. Then, there exists $\Delta_j \in R^{\Gamma}(\Delta_i)$ s.t. $(W^{\Gamma}, R^{\Gamma}, V^{\Gamma}, S^{\Gamma} \cup \{\Delta_j\}), \Delta_j \vDash \psi$. Now, using Lemma 3, we know that $(W^{\Gamma}, R^{\Gamma}, V^{\Gamma}, S^{\Gamma}), \Delta_j \vDash \psi[\mathfrak{p}/\mathfrak{p} \vee j]$. Note that $C(\langle \mathfrak{p} \rangle\psi) > C(\psi[\mathfrak{p}/\mathfrak{p} \vee j])$. Then, by induction hypothesis, it holds that $\psi[\mathfrak{p}/\mathfrak{p} \vee j] \in \Delta_j$. Thus, $@_i\langle \mathfrak{p} \rangle j \wedge @_j\psi[\mathfrak{p}/\mathfrak{p} \vee j] \in \Gamma$. By the axiom ($\mathsf{Paste}_{\langle \mathfrak{p} \rangle}$), it holds that $@_i\langle \mathfrak{p} \rangle\psi \in \Gamma$, which then gives us $\langle \mathfrak{p} \rangle\psi \in \Delta_i$, as desired.

For the direction from right to left, suppose that $\langle \mathfrak{p} \rangle\psi \in \Delta_i$. Then, $@_i\langle \mathfrak{p} \rangle\psi \in \Gamma$. Since Γ is $\mathfrak{p}$-pasted, there is a nominal $j \in \mathsf{Nom+}$ s.t. $@_i\langle \mathfrak{p} \rangle j \wedge @_j\psi[\mathfrak{p}/\mathfrak{p} \vee j] \in \Gamma$. Then, $\Delta_j \in R^{\Gamma}(\Delta_i)$ and $\psi[\mathfrak{p}/\mathfrak{p} \vee j] \in \Delta_j$. By induction hypothesis, $(W^{\Gamma}, R^{\Gamma}, V^{\Gamma}, S^{\Gamma}), \Delta_j \vDash \psi[\mathfrak{p}/\mathfrak{p} \vee j]$. Then by Lemma 3, $(W^{\Gamma}, R^{\Gamma}, V^{\Gamma}, S^{\Gamma} \cup \{\Delta_j\}), \Delta_j \vDash \psi$. Thus, $\mathfrak{M}^{\Gamma}, \Delta_i \vDash \langle \mathfrak{p} \rangle\psi$. This completes the proof.□

We now have enough background to show the following:

6 Note that $\mathfrak{p} \vee i := \neg(\neg\mathfrak{p} \wedge \neg i)$, and so $C(\varphi[\mathfrak{p}/\mathfrak{p} \vee i]) = C(\varphi) + 4 \times |\mathfrak{p}|$. Also, we assume that $\langle \mathfrak{p} \rangle$ occurs n times in φ; we have $C(\langle \mathfrak{p} \rangle\varphi) = 4 \times (|\mathfrak{p}| + 1) \times ((n + 1) + 1) + |\neg| + |\Diamond| + |@| + |\wedge| = C(\varphi) + 4 \times (|\mathfrak{p}| + 1)$. Therefore, $C(\langle \mathfrak{p} \rangle\varphi) > C(\varphi[\mathfrak{p}/\mathfrak{p} \vee i])$.

Theorem 4. **HMPML** *is strongly complete for the class of all frames, i.e., every* **HMPML**-*consistent set is satisfiable w.r.t. the class of all frames*.

Proof. Given an **HMPML**-consistent set Δ, Lemma 4 shows that we can extend it to a named, pasted and $\mathfrak{p}$-pasted **HMPML**-MCS $\Gamma \subseteq \mathcal{L}^{+}_{\mathsf{HM}}$. By Lemma 5, it holds that $\mathfrak{M}^{\Gamma}, \Gamma \vDash \Gamma$. Now, we define a new model $\mathfrak{M} = (W^{\Gamma}, R^{\Gamma}, V, S^{\Gamma})$, where $V = V^{\Gamma} \upharpoonright (\mathsf{Prop} \cup \mathsf{Nom})$ is obtained by restricting V^{Γ} to $\mathsf{Prop} \cup \mathsf{Nom}$. By Proposition 2, $\mathfrak{M}, \Gamma \vDash \Delta$. Thus, Δ is satisfiable. □

Now, let **HPML** be the proof system obtained by extending **HMPML** with a new axiom $\neg \mathfrak{p}$. We have the following result.

Theorem 5. **HPML** *is a sound and strongly complete calculus for* HPML.

Proof. The soundness is easy to see. The proof of completeness is the same as that for **HMPML**, and it is crucial to note that with the formula $\neg \mathfrak{p}$ as an axiom, the set S^{Γ} of the canonical model (Definition 4) is $\emptyset$. □

4.3 A calculus for HMPSL

As stated, HMPSL is a fragment of HMPML. In line with this, we will show that by using some provable formulas and derivable rules of **HMPML**, we can obtained a desired calculus, written **HMPSL**, for HMPSL. In this part, we present the details of **HMPSL**, and the next part will show a completeness result for the proof system.

Before introducing **HMPSL**, let us first note that some axioms and rules of **HMPML** are involved with formulas of the form $\varphi[\mathfrak{p}/\mathfrak{p} \vee j]$, e.g., (Memory) and ($\mathsf{Paste}_{\mathfrak{p}}$), and as stated, it is intended to capture the situation that a new state is added to the memory set. To axiomatize HMPSL, we also need a similar manipulation, but the situation for the new logic is more intricate, because of the fact that the meaning of $\langle t \rangle$ is richer than $\Diamond$. Let us now introduce the following syntactic translation for $\mathcal{L}_{\mathsf{HS}}$ that aims to handle the situation of memory expansion.

Definition 6. *Given a nominal* $i \in \mathsf{Nom}$, *the translation* $(\cdot)^{*}_{i} : \mathcal{L}_{\mathsf{HS}} \to \mathcal{L}_{\mathsf{HS}}$ *recursively as follows:*

$$(\mathfrak{p})^{*}_{i} \coloneqq p \qquad (j)^{*}_{i} \coloneqq j \qquad (\mathfrak{p})^{*}_{i} \coloneqq \mathfrak{p} \vee i \qquad (\neg\varphi)^{*}_{i} \coloneqq \neg(\varphi)^{*}_{i} \qquad (\varphi \wedge \psi)^{*}_{i} \coloneqq (\varphi)^{*}_{i} \wedge (\psi)^{*}_{i}$$

$$(@_{j}\, \varphi)^{*}_{i} \coloneqq @_{j}(\varphi)^{*}_{i} \qquad (\langle \mathfrak{p} \rangle \varphi)^{*}_{i} \coloneqq \langle \mathfrak{p} \rangle (\varphi)^{*}_{i} \qquad (\langle t \rangle \varphi)^{*}_{i} \coloneqq \langle t \rangle(\neg i \wedge (\varphi)^{*}_{i})$$

For instance, $(\langle t\rangle\langle t\rangle \mathfrak{p})^*_j$ is $\langle t\rangle(\neg j \wedge \langle t\rangle(\neg j \wedge (\mathfrak{p} \vee j)))$. W.r.t. the translation, the following holds:

Lemma 6. *Let* $\mathfrak{M} = (W, R, V, S)$ *be a model and* $w \in W$ *such that* $V(i) = w$. *Then for each* $v \in W$,

$$(W, R, V, S \cup \{w\}), v \vDash \varphi \textit{ iff } (W, R, V, S), v \vDash (\varphi)^*_i$$

Proof. We show this by induction on φ. We merely prove for the case that $\varphi \equiv \langle t\rangle\psi$, as all other cases are similar to those in the proof for Lemma 3.

Suppose $(W, R, V, S \cup \{w\}), v \vDash \langle t\rangle\psi$. Then there exists $u \in R(v)$ such that $u \notin S \cup \{w\}$ and $(W, R, V, S \cup \{w\}), u \vDash \psi$. By induction hypothesis, $\mathfrak{M}, u \vDash (\psi)^*_i$. Since $V(i) = w$ and $u \notin S \cup \{w\}$, it holds that $\mathfrak{M}, u \nvDash i$. Therefore, we have $\mathfrak{M}, v \vDash \langle t\rangle(\neg i \wedge (\psi)^*_i)$, i.e., $\mathfrak{M}, v \vDash (\langle t\rangle y)^*_i$.

For the converse direction, assume that $\mathfrak{M}, v \vDash (\langle t\rangle y)^*_i$, i.e., $\mathfrak{M}, v \vDash \langle t\rangle(\neg i \wedge (\psi)^*_i)$. Then, there exists $u \in R(v)$ s.t. $u \notin S$ and $(W, R, V, S), u \vDash \neg i \wedge (\psi)^*_i$. So, $u \neq w$, which entails $u \notin S \cup \{w\}$. By induction hypothesis, $(W, R, V, S \cup \{w\}), u \vDash \psi$. Hence, $(W, R, V, S \cup \{w\})$, $v \vDash \langle t\rangle\psi$. □

As an application of Lemmas 3 and 6, we have the following:

Corollary 1. *For any* $\varphi \in \mathcal{L}_{\mathsf{HM}}$, $@_j\varphi[\mathfrak{p} / \mathfrak{p} \vee j] \leftrightarrow @_j(\varphi)^*_j$ *is valid. Then, by the completeness of* **HMPML**, *it holds that* $\vdash_{\mathbf{HMPML}} @_j\varphi[\mathfrak{p} / \mathfrak{p} \vee j] \leftrightarrow @_j(\varphi)^*_j$.

Note that the equivalence holds with the understanding that $\langle t\rangle\varphi$ is just an abbreviation of formula $\Diamond(\varphi \wedge \neg \mathfrak{p})$, but it does not mean that the functions $(\cdot)^*_i$ are redundant; the functions are defined for the case that $\langle t\rangle\varphi$ is a primitive modality of HPSL that does not include the operator $\Diamond\varphi$.

Proposition 4 *In* **HMPML**, *the axioms in Table 2 are provable and the rules in the table are derivable*.

Proof. We state briefly the key reasons how we obtained the results, but skip the details to save space.

The cases for $(\mathsf{K}_{[t]})$, $(\mathsf{Dual}_{[t]})$, $(\mathsf{Back}_{[t]})$ and $(\mathsf{Nec}_{[t]})$ can be proved with the reasoning of propositional logic (as well as the definition of $\langle t\rangle\varphi$).

The formula $(\mathsf{Com}\langle \mathfrak{p}\rangle\langle t\rangle)$ can be proved directly with $(\mathsf{Com}\langle \mathfrak{p}\rangle\Diamond)$ and the definition of $\langle t\rangle\varphi$. For the formula $(\mathsf{Com}\langle t\rangle\langle \mathfrak{p}\rangle)$, one just needs to note that $@_i\langle t\rangle j \to @_i\Diamond j$ is provable.

Table 2. Axioms and rules for $\langle t\rangle\varphi$.

($K_{[t]}$)	$[t](\varphi \to \psi) \to ([t]\varphi \to [t]\psi)$
($Dual_{[t]}$)	$\langle t\rangle\varphi \leftrightarrow \neg[t]\neg\varphi$
($Back_{[t]}$)	$@_i\varphi \to [t]@_i\varphi$, where $i \in$ Nom.
($Memory^+$)	$@_i\langle \mathfrak{p} \rangle j \wedge @_j \varphi^*_j \to @_i\langle \mathfrak{p} \rangle\varphi$
($Com\langle \mathfrak{p} \rangle\langle t\rangle$)	$@_i\langle \mathfrak{p} \rangle j \wedge @_j (\varphi \wedge \neg \mathfrak{p}) \to @_i\langle t\rangle\varphi$
($Com\langle t\rangle\langle \mathfrak{p} \rangle$)	$@_i\langle t\rangle j \to @_i\langle \mathfrak{p} \rangle j$
($Nec_{[t]}$)	From φ, infer $[t]\varphi$.
($Paste_{\langle t\rangle}$)	From $@_i\langle t\rangle j \wedge @_j(\varphi \wedge \neg \mathfrak{p}) \to \psi$, infer $@_i\langle t\rangle\varphi \to \psi$, where j is new to φ, ψ, i.
($Paste^+_{\langle \mathfrak{p}\rangle}$)	From $@_i\langle \mathfrak{p} \rangle j \wedge @_j \varphi^*_j \to \psi$, infer $@i\langle \mathfrak{p} \rangle\varphi \to \psi$, where j is new to φ, ψ, i.

For the rule ($Paste_{\langle t\rangle}$), one can prove it easily with the rule ($Paste_{\Diamond}$) and the fact that the formula $@_i\Diamond j \wedge @_j(\varphi \wedge \neg \mathfrak{p}) \to @_i\langle t\rangle j \wedge @_j(\varphi \wedge \neg \mathfrak{p})$ is provable in **HMPML**.

Finally, for ($Memory^+$) and ($Paste^+_{\langle \mathfrak{p}\rangle}$), we can show them using the formula $@_j\varphi[\mathfrak{p}/\mathfrak{p} \vee j] \leftrightarrow @_j(\varphi)^*_j$ in Corollary 1 together with (Memory) and ($Paste_{\langle \mathfrak{p} \rangle}$), respectively. □

Now, we can show the details of **HMPSL**, which is obtained by modifying **HMPML** in Table 1 in the following manner:

(i) Remove the axioms and rules involving $\Diamond$, i.e., ($K_{\Box}$), ($Dual_{\Box}$), ($Back_{\Box}$), ($Nec_{\Box}$), ($Paste_{\Diamond}$), ($Com\langle \mathfrak{p} \rangle\Diamond$) and ($Com\Diamond\langle \mathfrak{p} \rangle$).

(ii) Remove (Memory) and ($Paste_{\langle \mathfrak{p} \rangle}$).

(iii) Add the axioms and rules in Table 2.

It is simple to check that the resulting calculus **HMPSL** is sound.

Theorem 7. *For all formula $\varphi \in \mathcal{L}_{HS}$, $\vdash_{\mathbf{HMPSL}} \varphi$ implies that φ is a validity of* HMPSL.

4.4 Completeness for HMPSL

We now turn to showing the strong completeness for **HMPSL**, and the strategy is similar to that for **HMPML**. Below are some preliminary notions.

Definition 7. *Let $\Gamma \subseteq \mathcal{L}_{HS}$ be a set of formulas.*

- *Γ is* $\langle t\rangle$-pasted, *if for each $@_i\langle t\rangle\varphi \in \Gamma$, there is some nominal j such that $@_i\langle t\rangle j \wedge @_j(\varphi \wedge \neg \mathfrak{p}) \in \Gamma$).*

- *Γ is* $\mathfrak{p}^+$-pasted, *if for each* $@_i\langle \mathfrak{p} \rangle\varphi \in \Gamma$*, there is some nominal j such that* $@_i\langle \mathfrak{p} \rangle j \wedge @_j(\varphi)^*_j \in \Gamma$.

Moreover, the notions of **HMPSL**-*consistency*, **HMPSL**-*inconsistency* and *maximal* **HMPSL**-*consistent set* (**HMPSL**-*MCS*) are defined in a similar way to those for **HMPML**. Also, similar to the case of $\mathcal{L}^+_{\mathsf{HM}}$, we can also extend the language $\mathcal{L}_{\mathsf{HS}}$ with new nominals Nom′, and we still write Nom+ for Nom ∪ Nom′ and $\mathcal{L}^+_{\mathsf{HS}}$ for the resulting language. Now, we have the following Lindenbaum-style lemma.

Lemma 7. *Any* **HMPSL**-*consistent set of* $\mathcal{L}_{\mathsf{HS}}$ *can be extended to a named*, ⟨*t*⟩-*pasted and* $\mathfrak{p}^+$-*pasted* **HMPSL**-*MCS of* $\mathcal{L}^+_{\mathsf{HS}}$.

Note that Proposition 3 also holds for **HMPSL**-MCSs. The definition for *induced canonical models* is the same as Definition 4, except that the **HMPSL**-MCSs Γ are assumed to be named, ⟨*t*⟩-pasted and $\mathfrak{p}^+$-pasted. However, for simplicity, when we mention the induced canonical models for HPSL, we will still refer to Definition 4. Let us introduce a new definition for the complexity of $\mathcal{L}_{\mathsf{HS}}$-formulas.

Definition 8. *The* complexity of $\mathcal{L}_{\mathsf{HS}}$-formulas *is defined as*

$$C(\varphi) = 4 \times (|\mathfrak{p}| + 1) \times (|\langle \mathfrak{p} \rangle| + 1) \times (|\langle t\rangle| + 1) + |\neg| + |@| + |\wedge|$$

where $|\circ|$ *for* $\circ \in \{\mathfrak{p}, \langle \mathfrak{p} \rangle, \langle t\rangle, \neg, @, \wedge\}$ *is the number of occurrences of* $\circ$ *in* φ.

The notion ensures that $c(\langle \mathfrak{p} \rangle\varphi) > c((\varphi)^*_i)$.[7] Now we can show the following *Truth Lemma*.

Lemma 8. *Let* $\varphi \in \mathcal{L}^+_{\mathsf{HS}}$ *and* $\Delta_i \in W^\Gamma$. *Then*

$$\mathfrak{M}^\Gamma, \Delta_i \vDash \varphi \textit{ iff } \varphi \in \Delta_i.$$

Proof. The proof proceeds by induction on $c(\varphi)$. We merely consider the case that $\varphi \equiv \langle t\rangle\psi$, and the other cases are similar to the proof for Lemma 5.

First, we assume that $\mathfrak{M}^\Gamma, \Delta_i \vDash \langle t\rangle\psi$. Then there exists $\Delta_j \in R^\Gamma(\Delta_i)$ s.t. $\Delta_j \notin S^\Gamma$ and $\mathfrak{M}^\Gamma, \Delta_j \vDash \psi$. By induction hypothesis, $\psi \in \Delta_j$. Note that $\mathfrak{M}^\Gamma, \Delta_j \vDash \neg\mathfrak{p}$, so we have $\neg\mathfrak{p} \in \Delta_j$ (this holds in the case for $\varphi \equiv \mathfrak{p}$, which is omitted here). Then, $\psi \wedge \neg\mathfrak{p} \in \Delta_j$ and so $@_j(\psi \wedge \neg\mathfrak{p}) \in \Gamma$.

7 One can check that $c((\varphi)^*_i) = c(\varphi) + 4 \times|\mathfrak{p}| + 2 \times|\langle t\rangle|$ (using again the facts that $\mathfrak{p} \vee i$ is the abbreviation of $\neg(\neg\mathfrak{p} \wedge\neg i)$) and $c(\langle \mathfrak{p} \rangle\varphi) = c(\varphi) + 4 \times(|\mathfrak{p}| + 1) \times(|\langle t\rangle| + 1) = c(\varphi) + 4 \times|\mathfrak{p}| \times|\langle t\rangle| + 4 \times|\mathfrak{p}| + 4 \times|\langle t\rangle| + 4$.

Also, by Definition 4, $@_i\langle \mathfrak{p} \rangle j \in \Gamma$. Thus $@_i\langle \mathfrak{p} \rangle j \wedge @_j(\psi \wedge \neg \mathfrak{p}) \in \Gamma$. Then by axiom (Com$\langle \mathfrak{p} \rangle\langle t\rangle$) and (MP), we have $@_i\langle t\rangle\psi \in \Gamma$, i.e., $\varphi \in \Delta_i$.

Next, suppose $\langle t\rangle\psi \in \Delta_i$. Then $@_i\langle t\rangle\psi \in \Gamma$. Since Γ used to induce the canonical model is assumed to be $\langle t\rangle$-pasted, there exists a nominal j s.t. $@_i\langle t\rangle j \wedge @_j(\psi \wedge \neg \mathfrak{p}) \in \Gamma$. It follows from $@_i\langle t\rangle j$ and the axiom (Com$\langle t\rangle\langle \mathfrak{p} \rangle$) that $@_i\langle \mathfrak{p} \rangle j \in \Gamma$. Then $\Delta_j \in R^\Gamma(\Delta_i)$. Also, from $@_j(\psi \wedge \neg \mathfrak{p}) \in \Gamma$ we know that $\psi \in \Delta_j$ and $\neg \mathfrak{p} \in \Delta_j$. Because of the latter, it holds that $\Delta_j \notin S^\Gamma$, and by induction hypothesis, the former gives us $\mathfrak{M}^\Gamma, \Delta_j \vDash \psi$. Thus, $\mathfrak{M}^\Gamma, \Delta_i \vDash \varphi$. □

Then by Lemma 8, we immediately have the following:

Theorem 8. **HMPSL** *is strongly complete for the class of frames.*

Finally, similar to the case of **HMPSL**, we write **HMPSL** for the proof system obtained by adding $\neg \mathfrak{p}$ as an axiom to **HPSL**, and the following indicates that it is a desired calculus for HPSL.

Theorem 9. **HPSL** *is a sound and strongly complete calculus for* HPSL.

5 Conclusion

In this article, we studied two logics for the poison game, PML and PSL, which are designed on the basis of the memory logic but are strictly weaker than the latter. As indicated in the work of Blando *et al*. (2020), PSL is strictly weaker than PML and the latter has an undecidable satisfiability problem. In this work, we offered the same answer to the satisfiability problem for PSL. Also, based on the complexity of the model-checking problems for PML and PSL, we showed that the logics cannot be translated in fixed-variable fragments of FOL. Moreover, motivated by the techniques developed in the work of Areces *et al*. (2012), we enhanced PML and PSL with formulas from hybrid logic, and then explored complete Hilbert-style proof systems for the resulting logics and the corresponding minimal logics. As we have seen, in line with the fact that hybrid PML is strictly stronger than the hybrid PSL, the proof system for the latter is also a 'fragment' of that for the former.

Before closing the paper, it is important to note that besides the poison game, there are many other graph games and logics having interesting interactions. For instance, van Benthem (2005); Aucher *et al*. (2015); Aucher *et al*. (2018) explore the sabotage games and the matching modal logic, in which a player can delete links in an arbitrary way. As illustrated in the works of Gierasimczuk *et al*. (2009); Baltag *et al*. (2019); Baltag *et al*. (2022), the sabotage-style link modification is also useful in characterizing many features in learning/teaching scenarios. Closely relevant to this, Li (2020) studies the case that links are removed locally according to certain properties expressed explicitly in the language proposed. Moreover, Thompson (2020) studies a dynamic logic of local fact changes that captures a class of graph games in which properties of vertices might be affected by other vertices. In addition, Li *et al*. (2021; 2023) explore the games of hide and seek (also known as cops and robbers) with a logical approach, which can model pursuit-evasion environments with players having their goals entangled. See also the works of Sano *et al*. (2024) and Chen and Li (2024) for further developments of the logic for the hide and seek game, and the work of Li *et al*. (2025) for the imperfect information setting. We refer to van Benthem and Liu (2020) for a broad program on this topic and to van Benthem and Liu (2025) for the latest developments.

Finally, let us end the paper with several further directions deserving to be explored. A natural next step is to identify the complexity of the model-checking problems for HPSL and HPML. Close to this, it is also important to know the exact complexity of the satisfiability problems for PSL and PML and to identify some non-trivial decidable fragments of the two logics (including both syntactic fragments and the restrictions to graphs with certain relations, e.g., reflexive relations and the more complicated mereological structures (Varzi, 2019)). Another direction is to explore the expressive power of HPML and HPSL, at the levels of both models and frames. Moreover, hybrid logics match well with the tableau techniques (Bolander and Blackburn, 2007; Indrzejczak and Zawidzki, 2013), and it would be interesting to study the desired tableau calculi for these logics, referring to the calculi for both hybrid logic and memory logic (Areces *et al*., 2009). Also, memory logic is closely related to the sabotage modal logic (Aucher *et al*., 2018) and the class of relation-changing logics (Areces *et al*., 2018), and in line with the strategy to axiomatize our logics and the standard memory logic (Areces *et al*., 2012), they are also axiomatized in the setting with hybrid formulas (Du and Chen, 2024; van Benthem *et al*., 2023). Notably, those logics develop various ways to update relations of models, which suggests that it would be interesting to consider other mechanisms of memorizing (as well as forgetting). In particular, it is meaningful to consider the memorizing and forgetting with an explicit definition, akin to the case of definable link deletion (Li, 2020), the logic of stepwise removal (van Benthem *et al*., 2022) and the dynamic-epistemic logic (van Ditmarsch *et al*., 2007; van Benthem, 2011), which would be useful in placing memory logic to a broader setting connecting different traditions.

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References

C. Areces, R. Fervari, G. Hoffmann, and M. Martel. (2018) Undecidability of relation-changing modal logics. In A. Madeira and M. Benevides, editors, *Proceedings of DaLÍ 2017*, volume 10669 of *LNCS*, 1–16. Springer.

C. Areces, D. Figueira, S. Figueira, and S. Mera. (2008) Expressive power and decidability for memory logics. In W. Hodges and R. de Queiroz, editors, *WoLLIC 2008*, volume 5110 of *LNCS*, 56–58. Springer.

C. Areces, D. Figueira, S. Figueira, and S. Mera. (2011) The expressive power of memory logics. *The Review of Symbolic Logic*, 4:290–318.

C. Areces, D. Figueira, D. Gorín, and S. Mera. (2009) Tableaux and model checking for memory logics. In M. Giese and A. Waaler, editors, *Automated Reasoning with Analytic Tableaux and Related Methods*, volume 5607 of *LNAI*, 47–61. Springer.

C. Areces, S. Figueira, and S. Mera. (2012) Completeness results for memory logics. *Annals of Pure and Applied Logic*, 163:961–972.

G. Aucher, J. van Benthem, and D. Grossi. (2015) Sabotage modal logic: Some model and proof theoretic aspects. In W. van der Hoek, W. Holliday, and W. Wang, editors, *Proceedings of LORI 2015*, volume 9394 of *LNCS*, 1–13. Springer.

G. Aucher, J. van Benthem, and D. Grossi. (2018) Modal logics of sabotage revisited. *Journal of Logic and Computation*, 28(2):269–303.

A. Baltag, D. Li, and M. Y. Pedersen. (2019) On the right path: A modal logic for supervised learning. In P. Blackburn, E. Lorini, and M. Guo, editors, *Proceedings of LORI 2019*, volume 11813 of *LNCS*, 1–14. Springer.

A. Baltag, D. Li, and M. Y. Pedersen. (2022) A modal logic for supervised learning. *Journal of Logic, Language and Information*, 31:213–234.

R. Berger. (1966) *The Undecidability of the Domino Problem*. AMS.

P. Blackburn, M. de Rijke, and Y. Venema. (2001) *Modal Logic*. Cambridge University Press.

P. Blackburn and J. Seligman. (1995) Hybrid languages. *Journal of Logic, Language and Information*, 4:251–272.

F. Z. Blando, K. Mierzewski, and C. Areces. (2020) The modal logics of the poison game. In F. Liu, H. Ono, and J. Yu, editors, *Knowledge, Proof and Dynamics. The Fourth Asian Workshop on Philosophical Logic*, 3–23. Springer.

T. Bolander and P. Blackburn. (2007) Termination for hybrid tableaus. *Journal of Logic and Computation*, 17:517–554.

Q. Chen and D. Li. (2024) Logic of the hide and seek game: characterization, axiomatization, decidability. In N. Gierasimczuk and F. R. Velázquez-Quesada, editors, *Proceedings of DaLÍ 2023*, volume 14401 of *LNCS*, 20–34. Springer, 2024.

P. Du and Q. Chen. (2024) Axiomatization of hybrid logic of link variations. In N. Gierasimczuk and F. R. Velázquez-Quesada, editors, *Proceedings of DaLÍ 2023*, volume 14401 of *LNCS*, 35–51. Springer.

P. Duchet and H. Meyniel. (1993) Kernels in directed graph: a poison game. *Discrete Mathematics*, 115:273–276.

N. Gierasimczuk, L. Kurzen, and F. R. Velázquez-Quesada. (2009) Learning and teaching as a game: a sabotage approach. In X. He, J. Horty, and E. Pacuit, editors, *Proceedings of LORI 2009*, volume 5834 of *LNCS*, 119–132. Springer.

D. Grossi and S. Rey. (2019a) Credulous acceptability, poison games and modal logic. In N. Agmon, M. E. Taylor, E. Elkind, and M. Veloso, editors, *Proceedings of AAMAS 2019*, 1994–1996.

D. Grossi and S. Rey. (2019b) Credulous acceptability, poison games and modal logic. http://arxiv.org/abs/1901.09180v2

A. Indrzejczak and M. Zawidzki. (2013) Decision procedures for some strong hybrid logics. *Logic and Logical Philosophy*, 22:389–409.

D. Li. (2020) Losing connection: the modal logic of definable link deletion. *Journal of Logic and Computation*, 30:715–743.

D. Li, S. Ghosh, and F. Liu. (2025) Reasoning under uncertainty in the game of Cops and Robbers. *Synthese*, https://doi.org/10.1007/s11229-025-05165-6

D. Li, S. Ghosh, F. Liu, and Y. Tu. (2021) On the subtle nature of a simple logic of the hide and seek game. In A. Silva, R. Wassermann, and R. de Queiroz, editors, *Proceedings of WoLLIC 2021*, volume 13038 of *LNCS*, 201–218. Springer.

D. Li, S. Ghosh, F. Liu, and Y. Tu. (2023) A simple logic of the hide and seek game. *Studia Logica*, 111:821–853.

K. Sano, F. Liu, and D. Li. (2024) Hybrid logic of the hide and seek game. *Studia Logica*, https://doi.org/10.1007/s11225-024-10149-7

B. ten Cate. (2005) *Model Theory for Extended Modal Languages*. PhD thesis, Institute for Logic, Language and Computation, University of Amsterdam, Amsterdam.

D. Thompson. (2020) Local fact change logic. In F. Liu, H. Ono, and J. Yu, editors, *Knowledge, Proof and Dynamics*, Logic in Asia: Studia Logica Library, 73–96. Springer.

J. van Benthem. (2005) An essay on sabotage and obstruction. In D. Hutter and W. Stephan, editors, *Mechanizing Mathematical Reasoning*, volume 2605 of *LNCS*, 268–276. Springer.

J. van Benthem. (2011) *Logical Dynamics of Information and Interaction*. Cambridge University Press.

J. van Benthem, L. Li, C. Shi, and H. Yin. (2023) Hybrid sabotage modal logic. *Journal of Logic and Computation*, 33(6):1216–1242.

J. van Benthem and F. Liu. (2020) Graph games and logic design. In F. Liu, H. Ono, and J. Yu, editors, *Knowledge, Proof and Dynamics*, 125–146. Springer.

J. van Benthem and F. Liu, editors. (2025) *Graph Games and Logic Design: Recent Developments and Future Directions* (To Appear). Springer.

J. van Benthem, K. Mierzewski, and F. Z. Blando. (2022) The modal logic of stepwise removal. *The Review of Symbolic Logic*, 15(1):36–63.

H. van Ditmarsch, W. van der Hoek, and B. Kooi. (2007) Dynamic Epistemic Logic, volume 337 of *Synthese Library*. Springer, 2007.

M. Vardi. (1995) On the complexity of bounded–variable queries. In *Proceedings of the ACM 14th Symposium on Principles of Database Systems*, 266–276.

A. Varzi. (2019) Mereology. In E. N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, Spring 2019 edition.

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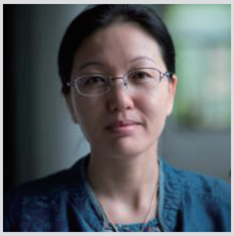

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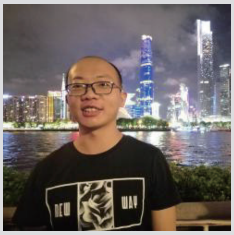

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